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Buildings and Non-positively Curved Polygons of Finite Groups

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Abstract. Let P be a non-positively curved polygon of finite groups. The group P acts on a contractible 2-complex X_P, and we prove that this complex is a building if and only if the links have (angular) diameter π. When P has zero group theoretic curvature, a geometric argument shows that the periodic apartments are dense in the set of all apartments.

Contents

1. Introduction

Theorem 4.2 (below) connects local combinatorial properties (the metric diameter of vertex links) with global geometric properties (being a building) of the developing complex of a non-positively curved polygon of groups. The technique of the proof of Theorem 4.2 is used to prove several results. Theorem 5.1 shows that

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for a non-positively curved polygon G of finite groups acting on a two-dimensional Euclidean building B, a positive power of almost every element of infinite order in G lies in a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of G. It follows (Corollary 5.3) that the set of periodic apartments is dense in the set of all apartments in the sense that any flat in B intersects some flat stabilized by a $\mathbb{Z} \oplus \mathbb{Z} < G$ in a ball of arbitrarily large radius.

1.1. **Related results.** Presenting these results at conferences, I was kindly directed to a number of related results in the literature. Roger Alperin and Jon Corson directed me to two papers of Jacque Tits [13, 12]. By different means and in more general circumstances, Theorem 4.2 was first proved by Tits; Tits's techniques are combinatorial in nature. Frédéric Paulin brought to my attention a paper of Sylvain Barré [1] where he gives a similar characterization of buildings in two dimensions.

2. Polygons of groups

A *polygon of groups* or *k-gon of groups* P is the colimit of a commutative diagram of groups and proper inclusions modelled on the poset of faces of the first barycentric subdivision of a polygonal 2-cell with $k \geq 3$ sides. For instance, a square of groups is the colimit of a diagram of the form

$$P = \operatorname{colim} \begin{pmatrix} V_1 & \longleftarrow & E_{1,2} & \longrightarrow & V_2 \\ \uparrow & & \uparrow & & \uparrow \\ E_{4,1} & \longleftarrow & F & \longrightarrow & E_{2,3} \\ \downarrow & & \downarrow & & \downarrow \\ V_4 & \longleftarrow & E_{3,4} & \longrightarrow & V_3 \end{pmatrix}$$

It is a convenient abuse of notation to confuse the diagram and the colimit, and for the purposes of this paper, such abuse is harmless. The V_i are *vertex groups*, the $E_{i,j}$ are *edge groups*, and F is the *face group*. The diagram defines projections

$$p_j : E_{i,j} \underset{F}{*} E_{j,k} \longrightarrow V_j.$$

If p_j is not injective, let w_j be a shortest non-trivial word in $\ker p_j$. The *angle* at V_j is defined to be $2\pi/|w_j|$ if p_j is not injective and 0 if p_j is injective.

Let G be a group and $\{G_\alpha\}$ be a set of subgroups. The *coset complex* of G relative to $\{G_\alpha\}$ is the nerve of the covering of G by cosets of the G_α. The canonical example of a coset complex is the tree for a free product of non-trivial finite groups: the tree upon which $F = G * H$ acts is the coset complex for F relative to $\{G, H\}$. If the vertex groups of a polygon of groups inject into the colimit, the polygon of groups is *developable*. The developing complex X_P for a developable polygon of groups is the coset complex of P relative to the set of vertex groups, i.e., the combinatorial cell complex whose vertices are the left cosets of P modulo the vertex groups, whose edges are the left cosets of P modulo the edge groups, and whose faces are the elements of P/F; the incidence relation is non-empty intersection, and P acts on the complex by multiplication.

Angles can also be (equivalently) defined combinatorially in terms of the coset complex of V_j relative to $\{E_{i,j}, E_{j,k}\}$; this coset graph is simply a quotient of the tree on which $E_{i,j} *_F E_{j,k}$ acts. If p_j fails to be injective, the angle at V_j is 2π divided by the girth[1] of the coset graph. If p_j is injective, the angle is defined to be zero. Supposing that P is developable, the coset graph of V_j relative to $\{E_{i,j}, E_{j,k}\}$ is the link of a vertex corresponding to a coset of V_j. With this in mind, the *link* of a pair of distinct proper subgroups $H, K < G$, abbreviated **Lk** $(H \hookrightarrow G \hookleftarrow K)$, is the coset complex for G relative to $\{H, K\}$. When the angle is non-zero, the link carries a natural metric such that the girth of the graph is 2π.

A k-gon of groups is *non-positively curved* if the sum of the angles at the vertex groups is no more than $(k-2)\pi$; note that the definition of polygon of groups guarantees that the angle at a vertex of a polygon of groups is no larger than $\pi/2$, so every k-gon of groups for $k \geq 4$ is non-positively curved. Gersten and Stallings [11] proved:

Theorem 2.1 (Gersten-Stallings). *Let P be a non-positively curved polygon of groups. Then,*

1. *P is developable.*
2. *The developing complex X_P is contractible and carries a* CAT(0) *or* CAT(−1) *metric.*

The metric on X_P is obtained by equivariantly metrizing each 2-cell as though it were a polygon in $\mathbb{E}^2$ or $\mathbb{H}^2$ with geodesic boundary and geometric angles corresponding to the angles at the vertex groups of P.

3. Links of diameter π

The first essential ingredient is a characterization of vertex links whose metric diameter is π.

3.1. **Diameter, girth, and completing partial circuits.** Let Γ be a finite, connected bipartite graph with no vertices of valence one; the bipartition implies that the girth ℓ of Γ is even. Use a path metric that assigns each edge equal length to metrize Γ so that Γ has metric girth 2π. The *angle* associated to Γ, $\angle\Gamma = 2\pi/\ell$ is well-defined (namely the length of an edge). The *diameter* of Γ, denoted diam(Γ) is the maximum distance between any two vertices. Note that diam$(\Gamma) \geq \pi$, because the shortest circuit in Γ has length 2π.

The situation where Γ has (minimal) diameter π can be characterized in terms of (simplicial) rays of length $\pi + \angle\Gamma$.

Proposition 3.1. *Let Γ be a finite, connected bipartite graph with with no vertices of valence one. Then, diam$(\Gamma) = \pi$ if and only if every simplicial ray α of length $\pi + \angle\Gamma$ is part of a unique circuit of length 2π.*

Proof. Suppose that the diameter of Γ is π. Then, there is a geodesic segment β of length less than or equal to π connecting the endpoints of α. If the length of β is π, then the circuit $\alpha \cup \beta$ contains an odd number of edges (which is impossible in a bipartite graph). If β is shorter than $\pi - \angle\Gamma$, then the circuit $\alpha \cup \beta$ has length shorter than 2π, contradicting the assumed girth of Γ. Thus, the length of β is $\pi - \angle\Gamma$ as claimed. Conversely, if Γ contains a geodesic of length strictly larger than π, then that geodesic is not part of any circuit of length 2π.

[1]The *girth* of a graph is the length of the shortest non-trivial circuit.

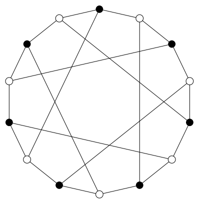

FIGURE 1. The link of $\left(\langle a\rangle \hookrightarrow \left\langle a,b:\ a^3, ab\bar{a}=b^2\right\rangle \hookleftarrow \langle ab\rangle\right)$ has angle $\frac{\pi}{3}$ and diameter π.

To establish uniqueness, observe that the intersection of two circuits γ and γ' of length 2π is a geodesic segment; otherwise,

$$(\gamma \cup \gamma') \setminus (\gamma \cap \gamma')$$

would contain a circuit of length $2\pi - 2\angle\Gamma$. □

Having diameter π has strong consequences for the structure of Γ. For example, a deep theorem of Walter Feit and Graham Higman [6] states that if Γ is finite and the valence of each vertex is at least three, then

$$\operatorname{diam}(\Gamma) = \pi \implies \angle\Gamma \in \left\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{8}\right\}.$$

The statement of the theorem also gives algebraic relations between the valences of the two types of vertices. Bipartite graphs with girth $2m$, diameter m (each edge has length one), and no vertices of valence one are called *generalized m-gons* in the literature. See, e.g., Ronan [10, pp. 28–30,36–38].

For present purposes, Γ will be the link of a vertex in the developing complex for some polygon of finite groups, in which case the vertex stabilizer G will act transitively and without inversion on the edges of Γ. It is convenient to be able to check the diameter of Γ with knowledge only of G. The following proposition provides one method.

Proposition 3.2. *Let G be a group acting on a finite bipartite graph Γ so that Γ/G is a single edge. Suppose diam$(\Gamma) = \pi$, $\angle\Gamma = \frac{\pi}{m}$, and let the edge $e = [u,v]$ be a fundamental domain for the action of G on Γ. Then, G is a quotient of $G_u *_{G_e} G_v$ and:*

- $G = \overbrace{G_uG_vG_uG_v\ldots}^{m\ factors}.$
- *The order of G is $|G_e|$ times the volume (in edges) of a ball of radius π in the tree on which $G_u *_{G_e} G_v$ acts.*

Proof. Consider the universal cover $\tilde{\Gamma}$ of Γ. The diameter of Γ is π and the girth of Γ is 2π, so the number of edges in a ball of radius π (about a vertex) in $\tilde{\Gamma}$ equals the number of edges in Γ. The first statement follows from Bass-Serre theory, and the second statement follows from the fact that the number of elements in G is $|G_e|$ times the number of edges in Γ. □

Corollary 3.3 (Uniqueness of F_{21})**.** *The Frobenius group of order* 21,

$$F_{21} = \left\langle x,\, y:\ x^3,\, y^3,\, xy\bar{x}\bar{y}x\bar{y}\right\rangle \tag{1}$$

and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ are the only finite groups generated by two $\mathbb{Z}_3$ subgroups forming a link of diameter π.

Proof. If two distinct $\mathbb{Z}_3$ subgroups form a link of diameter π and angle $\frac{\pi}{2}$ within a group G, then Proposition 3.2 implies that G has order nine. the cylic group of order nine does not contain two distinct cyclic subgroups of order three, so $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is the only possiblity. If two distinct subgroups of order 3 form a link of diameter π and angle $\frac{\pi}{3}$ within a group G, then Proposition 3.2 implies that G has order 21. The Frobenius group of order 21 is the only possibility for G, as the other group of order 21 is abelian and any two subgroups of an abelian group form an angle no smaller than $\frac{\pi}{2}$. (The link of $(\mathbb{Z}_3 \hookrightarrow F_{21} \hookleftarrow \mathbb{Z}_3)$ appears in Figure 1.) The Sylow theorems and some straightforward observations are sufficient to rule out groups of orders 45, 189, and 765 (corresponding to the possible angles of $\pi/4$, $\pi/6$, and $\pi/8$) and the Feit-Higman Theorem guarantees that there exist no groups G (such that the link would have diameter π) for angles smaller than $\frac{\pi}{8}$; so F_{21} and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ are the only finite groups where two $\mathbb{Z}_3$ subgroups can form a link of diameter π. $\square$

If the restriction on the diameter of the link is removed, then the possibilities vary widely. For example, consider the following pair of subgroups of a group of order 360:

$$\big(\langle x\rangle \hookrightarrow \big\langle x,\, y : x^3,\, y^3,\, xyxyx = yxyxy\big\rangle \hookleftarrow \langle y\rangle\big)\,. \tag{2}$$

The two specified $\mathbb{Z}_3$ subgroups form an angle of $\frac{\pi}{5}$, so the corollary implies that the diameter of the corresponding link is greater than π. (The diameter is $\frac{11\pi}{5}$.) Equivalently, the order of the group is 360, significantly larger than the 189 required for the link to have diameter π.

3.2. **Expanding planar disks.** Let P be a non-positively curved polygon of groups such the vertex links all have diameter π. A *planar disk* (respectively, a *plane*) D in X_P is a subcomplex which is convex, homeomorphic to a closed disk in $\mathbb{R}^2$ (respectively, to $\mathbb{R}^2$) and such that the total angular measure around an interior vertex is 2π. For example, in the case of a square of groups with all angles $\frac{\pi}{2}$, a planar disk would be an $n \times m$ rectangular grid of squares meeting four per interior vertex. For any subgraph $L \subset \mathbf{Lk}\, v$, the *cellular hull* of L is the corresponding subcomplex of $\mathbf{St}\, v$.

The lemma and propositions which follow describe the connection between link diameter π and planar subspaces of the developing complex X_P.

Lemma 3.4. *Let P be a non-positively curved polygon of groups with links of diameter π. Given any convex planar disk $\Delta \subset X_P$, there exists a convex planar disk $\Delta' \supset \Delta$ such that $\partial\Delta \subset \operatorname{int}\Delta'$.*

Proof. The idea of the proof is depicted in Figure 2 for the case of a $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ Euclidean triangle of groups. Any planar disk Δ is the union of planar neighborhoods of the interior vertices and planar neighborhoods of edges, so enlarging Δ is equivalent to choosing a compatible set of planar neighborhoods of the vertices of $\partial\Delta$.

Set $\Delta' = \Delta$ and choose a vertex $u \in \partial\Delta$ such that the angle of $\partial\Delta$ at u is less than π. Let u' be the next vertex of $\partial\Delta$ in counterclockwise order from u and choose a minimal circuit γ of $\mathbf{Lk}\, u'$ that contains $\mathbf{Lk}_{\Delta} u'$; add the cellular hull of γ to Δ'.

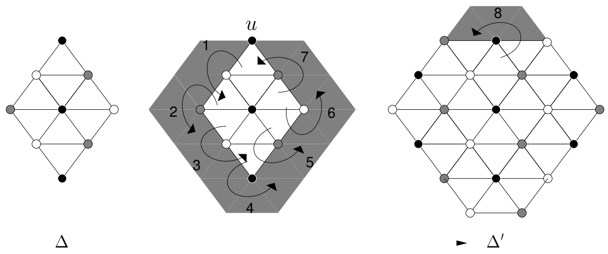

FIGURE 2. "Proof" of Lemma 3.4

Fix an orientation on Δ and let u'' be the next vertex of $\partial\Delta$ in counterclockwise order from u'. The angle of $\partial\Delta$ at u'' was at most π, so $\mathbf{Lk}_{\Delta'}u''$ is a segment of length less than or equal to $\pi + \angle\mathbf{Lk}\, v''$. Choose a compatible minimal circuit $\gamma' \subset \mathbf{Lk}\, u''$ and add the cellular hull of γ' to Δ'. Proceed in this fashion around $\partial\Delta$ until it is necessary to adjoin a planar neighborhood of u. The vertex u was chosen so that the angle of $\partial\Delta'$ at u would be π or less at this stage, so there is a minimal circuit in $\mathbf{Lk}\, u$ containing $\mathbf{Lk}_{\Delta'}u$; add the corresponding planar neighborhood to Δ'.

The only possible complication at this point is caused by *orphaned vertices* as shown in Figure 3. Three adjacent 2-cells around a $\frac{\pi}{2}$ vertex are not a convex configuration for a link of girth 2π. At each orphaned vertex, there exist unique cells to complete planar neighborhoods of the orphaned vertices, and once these cells are added, an interior angle of $\partial\Delta'$ is no more than twice one of the angles of P. Thus, Δ' is the desired convex planar disk. (Note that a local embedding of a planar disk into the developing complex must be an embedding, as otherwise there would be a closed geodesic.) $\square$

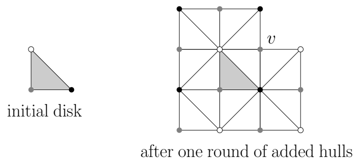

FIGURE 3. The disk Δ' with an orphaned vertex v.

If all vertex links have diameter π, this lemma can be used to isometrically embed a plane in the developing complex by successively expanding an embedded convex cellular disk. If the diameter of one of the links is larger than π, it is possible that the process of "growing" a plane by successive applications of Lemma 3.4 stalls at some point. The following proposition will prove useful later.

Proposition 3.5. *Let P be a non-positively curved polygon of finite groups with a vertex link L with diam$(L) > \pi$. There exist a pair of 2-cells in X_P which can not both lie in a single plane.*

Proof. There is a geodesic in L with length $\pi + \angle L$, so the initial and terminal edges can not both be on a minimal circuit of L, i.e., the corresponding two 2-cells can not both lie in any planar subset of X_P. □

4. Buildings

4.1. **Diameter π and buildings.** After defining buildings, we prove one of the main theorems.

A *Coxeter group* is a group with a presentation of the form

$$\left\langle x_1, \ldots, x_k : {x_1}^2, \ldots, {x_k}^2, (x_1x_2)^{m_{1,2}}, \ldots, (x_ix_j)^{m_{i,j}}, \ldots, (x_kx_{k-1})^{m_{k,k-1}} \right\rangle \tag{3}$$

subject to the requirements that the $m_{i,j}$ are from a symmetric matrix with coefficients in $\mathbb{Z}^+ \cup \{\infty\}$ and 1's down the diagonal. When the order of x_ix_j is "∞", the relation is omitted.

For any k-gon P in the 2-sphere S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$ with interior angles $\pi/\alpha_1, \ldots, \pi/\alpha_k$, there is an associated *Coxeter group*

$$\Delta(\alpha) = \left\langle x_1, \ldots, x_k : {x_1}^2, \ldots, {x_k}^2, (x_1x_2)^{\alpha_1}, \ldots, (x_ix_{i+1})^{\alpha_i}, \ldots, (x_kx_1)^{\alpha_k} \right\rangle, \tag{4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\alpha_i \geq 2$. In the context of polygons of groups, the Coxeter group $\Delta(\alpha)$ is a polygon of finite groups with $\mathbb{Z}_2$ edge groups and dihedral vertex groups, and the *Coxeter complex* of $\Delta(\alpha)$ is $X_{\Delta(\alpha)}$. In the study of two-dimensional buildings, only the Coxeter complexes for planar reflection groups are necessary, so the technicalities of a more general definition can be sidestepped. Ken Brown's book on buildings ([3, ch. III]) contains a general discussion.

Definition 4.1. A cell complex X is a *building* if

1. X is the union of a set of *apartments*, i.e., subcomplexes, isomorphic to $X_{\Delta(\alpha)}$ for some planar polygon P.
2. Given any pair of cells $\{\sigma, \tau\}$, there is an apartment $A \supset (\sigma \cup \tau)$.
3. If σ and τ are two top-dimensional cells of X contained in the intersection of two apartments A and B, then there is a cellular isomorphism $A \longrightarrow B$ fixing σ and τ.

These conditions guarantee that the apartments of X are all the same Coxeter complex. Spherical buildings and Euclidean buildings (also called *affine buildings* in the literature) buildings, named for the structure of their underlying Coxeter complexes, are well-known for their applications in the study of Lie groups and algebraic number theory. (See, e.g., Ken Brown's book [3, ch. VI–VII].) The theory of hyperbolic buildings, i.e., buildings in which each apartment is an $\mathbb{H}^n$ tessellated by the action of a hyperbolic reflection group, is less well developed. (See, e.g., Gaboriau and Paulin [7].)

Theorem 4.2 (J. Tits [13, 12])**.** *Let P be a non-positively curved polygon of finite groups. The vertex links all have diameter π if and only if X_P is a building.*

Proof. We verify the conditions in Definition 4.1.

Let $\{P_\alpha\}$ be the set of all planes in X_P. We take these to be the apartments of X_P. The first condition follows from Lemma 3.4, as any 2-cell is trivially a convex, isometrically embedded cellular disk.

The metric 2-complex X_P is geodesic, so the centers of any two 2-cells σ and τ can be connected with a geodesic segment L. The diameter of the link is the minimal π, so if ℓ runs through a vertex, its points of entry and exit form an angle of π. It follows from the ideas of the proof Lemma 3.4 that a convex cellular disk containing $\sigma \cup \tau \cup \ell$ can be created by growing a disk around σ so that each successive expansion contains more of ℓ. This process might require adding a triangle in the middle of a boundary segment of a disk and then tiling along a boundary segment in two directions, but the program of Lemma 3.4 is unaffected: this procedure still tiles-in the boundary segment of the disk in question, and the other boundary segments can be dealt with in counter-clockwise order as in the Lemma. This done, repeated application of Lemma 3.4 provides a family of embedded cellular disks which is strictly increasing with respect to inclusion; the union of this family is the desired plane.

Let P_i and P_j be such that $\sigma \cup \tau \subset P_i \cap P_j$. The intersection of planar subspaces is planar, so $\Delta = P_i \cap P_j$ is a possibly unbounded convex planar disk. The planes P_i and P_j are simply expansions of Δ in the spirit of Lemma 3.4, so the desired isomorphism is simply the limit of the isomorphisms between the incremental expansions of Δ as subsets of P_i and P_j.

The reverse implication follows from Proposition 3.5. □

4.2. **Squares of groups and tree $\times$ tree.** The following proposition illustrates an application of Theorem 4.2 to squares of groups.

Proposition 4.3. *Let S be a non-positively curved 4-gon of finite groups. The following are equivalent:*

1. *The vertex links of S are connected with diameter π and angle $\pi/2$.*
2. *The vertex links of S are complete bipartite graphs.*
3. *X_S is the product of two locally finite, uniform bipartite trees.*
4. *The order of each vertex group is the product of the orders of the incident edge groups divided by the order of the face group.*

Proof. The product of buildings is again a building, so $3 \Longrightarrow 1$ is immediate. The cells of a Euclidean building are either simplices or products of simplices, and when the cells are the product of lower-dimensional simplices, the building decomposes as a product of lower-dimensional buildings. (See [10, p. 110].) In this case, the cells of the building X_S are the product of two 1-simplices and X_S decomposes as the product of two 1-dimensional Euclidean buildings, i.e., trees. The local finiteness of X_S implies that the trees have finite valence, and thus $1 \Longrightarrow 3$. The equivalence of 1 and 2 is elementary, and the equivalence of 1 and 4 follows from the order computation in Proposition 3.2. □

The tree$\times$tree developing complexes have a particularly simple topological structure, but the polygons of groups acting on these complexes exhibit widely varied behavior. C. Y. Tang and G. Kim [9] have proven residual finiteness and subgroup-separability results for squares of nilpotent groups with cyclic edge groups and trivial face group, and Dani Wise [14] has produced non-residually finite flat squares of finite p-groups with developing complex tree $\times$ tree. Marc Burger and Shahar Mozes [5] have produced finitely presented infinite simple groups acting on tree$\times$tree.

5. Two-dimensional Euclidean buildings and $\mathbb{Z} \oplus \mathbb{Z}$'s

5.1. **Flat polygons of finite groups acting on Euclidean buildings.** A polygon of groups is *flat* if it has the angles of a Eulcidean polygon. Let P be a flat polygon of finite groups with X_P a building. To this point, we have been concerned with the existence and number of planes in X_P. Using a number of techniques inspired by Lemma 3.4, we now focus on $\mathbb{Z} \oplus \mathbb{Z}$ subgroups of P.

We begin with a variation on the idea of Lemma 3.4. Let w be an element of P. The vertex links of a building are necessarily connected (by condition 2 in Definition 4.1), so the edge groups generate P and w can be expressed as a word in elements of the edge groups in many different ways. Fix an initial 2-cell $\sigma \subset X_P$. A *row* of 2-cells in X_P is the area between two parallel cellular geodesic segments at minimum distance from each other. A *zig* in X_P is the union of two rows where the parallel cellular geodesics meet each other at the maximum angle, e.g., $\pi - \frac{\pi}{6}$ for a $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ triangle. The cells σ and $w \cdot \sigma$ both lie in some plane $Q \subset X_P$ (condition 2 in Definition 4.1), and σ and $w \cdot \sigma$ are connected by either a zig or a row lying entirely on Q. The element w is *straight* if σ and $w \cdot \sigma$ lie in some row and *bent* otherwise.

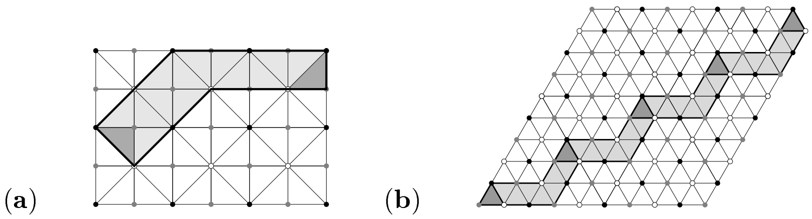

FIGURE 4. (**a**) A zig connecting two cells (shaded darker) for a $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ triangle. (**b**) a partial sequence of zigs for a $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ triangle with the w-translates of σ shaded.

A pair of two cells on a plane Q have the same *relative orientation* if one is the image of the other under a translation acting on the induced tiling of Q and this translation preserves stabilizer types. A nontrivial element w is a *translation* if w and $w \cdot \sigma$ have the same relative orientation in some (and thus any) plane. A pair of 2-cells occupies a finite set of relative orientations, so given an element w of infinite order, there exists a number n such that w^n is a translation.

Bent translations have non-trivial centralizers:

Theorem 5.1. *Let P be a flat polygon of finite groups with X_P a Euclidean building. For any bent element w of infinite order, there exists an element v and $n \in \mathbb{Z}^+$ such that $\langle w^n, v\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Equivalently, any bent translation lies in a $\mathbb{Z} \oplus \mathbb{Z} < P$ stabilizing a plane in X_P.*

Proof. Suppose that w is a bent translation, fix a zig representing it, a two-cell σ, and consider the subset $Z_0(w) \subset X_P$ which is the union of the zigs connecting $w^i \cdot \sigma$ with $w^{i+1} \cdot \sigma$ for $i \in \mathbb{Z}$. The boundary of $Z_0(w)$ is composed of cellular geodesics, and the angles at the boundary vertices of $Z_0(w)$ are π, $\pi + \epsilon$, or $\pi - \epsilon$ for ϵ the smallest angle of P. Select a vertex v with angle $\pi + \epsilon$, and complete the neighorhood of v in $Z_0(w)$ to a flat disk by adding 2-cells. Using the ideas of the proof of Lemma 3.4, the consequences of this addition give a new sequence of zigs

$Z_1(w)$ parallel to $Z_0(w)$ as in Figure 5. Observe that $Z_0(w)$ and $Z_1(w)$ are the same shape by construction.

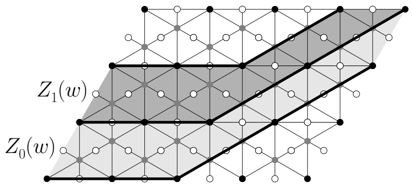

FIGURE 5. A part of $Z_0(w)$ and $Z_1(w)$ for a $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ triangle

By selecting two cells in the same relative positions in $Z_1(w)$ as σ and $w \cdot \sigma$ occupy in $Z_0(w)$, the new sequence of zigs $Z_1(w)$ represents a translation $w_1 \in P$. The translation w_1 is determined by w and has the same format and length as a word in the generators of the edge groups of P.

In the same fashion, $Z_1(w)$ uniquely determines a sequence of zigs $Z_2(w)$ with an associated word w_2 and so on, giving a sequence $Z_i(w)$ and words w_i for $i \in \mathbb{Z}^+$. By a mathematical headstand, the process is valid for negative indices as well. The assumed finiteness of the edge groups implies that there are only finitely many words of the same length and format as w. Each element is determined by the preceding element (or equivalently, the subsequent element) in the bi-infinite sequence $\ldots, w_{-1}, w, w_1, \ldots$, so every element which occurs once occurs infinitely often.

Because w occurs twice, say as w_0 and w_n, the uniform periodicity of the collections of zigs gives a "vertical" word v such that $wvw^{-1}v^{-1}$ fixes σ.

An elementary lemma completes the proof.

Lemma 5.2. *Let x and y be two elements of infinite order in a group G and suppose that all of the elements $\{[x, y^k] : k \in \mathbb{Z}^+\}$ lie in a finite subset of G. Then, there exists $n \in \mathbb{Z}^+$ such that $[x, y^n] = 1$.*

Proof. Infinitely many $[x, y^n]$ lie in a finite subset of G, so some two must be equal, say $[x, y^j] = [x, y^k]$ with $j \neq k$. Then,

$$\begin{aligned} xy^j\bar{x}\bar{y}^j &= xy^k\bar{x}\bar{y}^k \\ y^{j-k}\bar{x}y^{k-j} &= \bar{x}. \end{aligned} \tag{5}$$

and $[x, y^{k-j}] = 1$ as desired. □

The proof of Theorem 5.1 is complete. □

A plane in X_P is *periodic* if it is stabilized by a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of P.

Corollary 5.3. *For a flat polygon of finite groups P with links of diameter π, any planar disk in X_P is contained in a periodic plane, i.e., the set of periodic planes is a system of apartments for X_P.*

Proof. Given any planar disk $\Delta \subset X_P$, the disk can be enlarged until it is a parallelogram with a pair of zigs for boundaries; this is equivalent to the parallelogram having the sharpest possible angles at two corners. Carrying out the construction of Theorem 5.1 with either of the associated words (which both represent the same element) gives a plane $Q \supset \Delta$ with a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of P stabilizing Q. □

Equivalently,

Corollary 5.4. *Every plane in X_P is the limit of a sequence of periodic planes.*

Proof. Let Q be a plane in X_P and let $\{\Delta_i\}$ be a sequence of compact, convex cellular subsets of Q with $\Delta_i \subset \Delta_{i+1}$ and $Q = \bigcup \Delta_i$. Corollary 5.3 implies that there exists a sequence $\{Q_i\}$ of periodic planes with $\Delta_i \subset (Q_i \cap Q)$ and

$$\bigcup (Q_i \cap Q) = Q.$$

□

It would be interesting to know whether similar results hold for hyperbolic buildings:

Question 5.5. In a cocompact 2-dimensional hyperbolic building, is the set of periodic (under the actions of hyperbolic surface groups) apartments dense? Do the periodic apartments form a system of apartments for the building?

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