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A Correspondence for the Generalized Hecke Algebra of the Metaplectic Cover $\overline{SL(2,F)}$, F p-adic

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Abstract. We prove, using a technique developed for $GL(n)$ in Howe and Moy [H], a bijection between generalized Hecke algebras of $G = SL(2,F)$ over a p-adic field and those of its n-fold metaplectic cover $\overline{G}$. This result implies that there is a canonical correspondence between irreducible admissible representations of G and genuine irreducible admissible representations of $\overline{G}$ of "sufficiently large level" (depending on n, p).

Contents

Let F be a p-adic field of characteristic 0, with uniformizer π, ring of integers $\mathcal{O}$, and let q denote the cardinality of the residue field. Let $\mu_n(F)$ denote the group of n^{th} roots of unity in F and assume $|\mu_n(F)| = n$. We will identify $\mu_n(F)$ with a subgroup of $\mathbb{C}^\times$ (via some fixed isomorphism $\theta : \mu_n(F) \to \mu_n(\mathbb{C})$). Write $\mu_n = \mu_n(F)$. Let $G = SL(2,F)$ and let $\mathfrak{g}$ denotes its Lie algebra. Let $\overline{G} = \overline{SL(2,F)}$ denote the n-fold metaplectic cover defined by the cocycle

$$\beta(g_1, g_2) = \left(\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{x(g_2)}\right),$$

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for $g_1, g_2 \in G$, where

$$x\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{otherwise} \end{cases}$$

and where $(\ ,\ )$ is the n^{th} power Hilbert symbol. By §1 of Kubota [K], β is a 2-cocycle (called a "factor set" in [K]) which is trivial near the identity in $G \times G$, so $\overline{G}$ is a topological covering group of G. We denote elements of $\overline{G}$ by $\overline{x} = (x, \zeta)$, $x \in G,\ \zeta \in \mu_n$, and multiplication by

$$\overline{x}_1\overline{x}_2 = (x_1, \zeta_1)(x_2, \zeta_2) = (x_1x_2, \beta(x_1, x_2)\zeta_1\zeta_2).$$

Let $\rho : \overline{G} \to G$ denote the natural surjection. If H is any subgroup of G then we define $\overline{H} = \rho^{-1}(H)$. We choose a Haar measure on G, to be normalized below, and a Haar measure on μ_n normalized so that $\mathrm{vol}(\mu_n) = 1$. Let the Haar measure on $\overline{G}$ be the product measure.

We prove, using a technique developed for $GL(n)$ in [H], a bijection between generalized Hecke algebras of G and those of $\overline{G}$. A precise statement will be given in Theorem 11. This result implies that there is a canonical one-to-one correspondence between irreducible admissible representations of G and genuine irreducible admissible representations of $\overline{G}$ of "sufficiently large level" (depending on n, p). For the precise statement, see Corollary 12.

The study of genuine admissible representations on p-adic groups has its origins in the book [G] by S. Gelbart. Flicker [F] determined a comparison of Hecke algebras of *smooth functions* on $GL(2, F)$ and $\overline{GL(2, F)}$ using the matching of orbital integrals. This was then used, with the Arthur-Selberg trace formula, to establish a global "Shimura correspondence". Neither of these two works used the Hecke algebras considered here. G. Savin [S] has given an isomorphism between the Iwahori-Hecke algebras in the case when p is relatively prime to $2n$ and G is any reductive p-adic group which splits over $\mathcal{O}$. This isomorphism generalized a result of Flicker and Kazhdan [FK] who considered the special case of $GL(r)$. (Savin's result is discussed briefly in §6 below.) The works Rallis-Schiffmann [RS] and Schultz [Sch] also dealt with a correspondence between representations on $\overline{G}$ and on G (or a closely related group), though from a different perspective and using different methods. Our hope is that the result in the present paper will form one small step, along with Waldspurger's multiplicity one theorem for $\overline{SL(2, F)}$ [W] and the Arthur-Selberg trace formula, in an eventual proof Labesse's muliplicity one conjecture for $SL(2)$.

1. Some notation

Let

$$s_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & -1 \\ \pi & 0 \end{pmatrix}, \quad t_0 = \begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix}.$$

Note $t_0 = s_1^{-1}s_2$, $t \notin G$ and $ts_1t^{-1} = s_2$.

Let

$$u_+(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad u_-(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad h(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}.$$

Let

$$B = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} \right\} \cap G$$

denote the Iwawori subgroup,

$$\mathfrak{b} = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} \right\} \cap \mathfrak{g}$$

denote its Lie algebra, and let

$$B_k = \left(I + t^{2k} \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} \right) \cap G$$

denote a k^{th} filtration subgroup. Let

$$\mathfrak{b}_k = \left\{ \begin{pmatrix} a & b \\ \pi c & -a \end{pmatrix} \;\middle|\; a, b, c \in \pi^k \mathcal{O} \right\}$$

denote the Lie algebra of B_k.

Definition 1. Let H denote a locally compact unimodular group and let K denote an open compact subgroup of H with Haar measure on H normalized so that K has measure one. Let $\mathcal{H}(H//K)$ denote the space of functions on H which are locally constant, compactly supported, and bi-K-invariant. This forms an algebra under convolution,

$$(f * g)(x) = \int_H f(y) g(y^{-1}x) dy,$$

called the *generalized Hecke algebra.*

We shall study the algebra $\mathcal{H}(G//B_k)$ in Section 3.

Recall that a 2-cocycle $\alpha : H \times H \to \mu_n$ of a group H is *trivial* if there is a function $s : H \to \mu_n$ (called a *splitting*) such that $\alpha(h_1, h_2) = s(h_1)s(h_2)s(h_1h_2)^{-1}$, for $h_1, h_2 \in H$. In this case the cover of H defined by α is said to *split.*

Recall that a complex-valued function f on $\overline{G}$ is called *genuine* if $f(g, \zeta) = \theta(\zeta) f(g, 1)$, for $g \in G$ and $\zeta \in \mu_n$. Here θ is a fixed isomorphism $\theta : \mu_n \to \mu_n(\mathbb{C})$. It being agreed that we may identify μ_n and $\mu_n(\mathbb{C})$, we shall, *from this point on, drop the θ from the notation.*

Definition 2. Assume that $k > 0$ is chosen so large that $\overline{B}_k$ splits. Assume that the Haar measure on $\overline{G}$ is normalized so that $\overline{B_k}$ has measure 1. Let $\mathcal{H}(\overline{G}//B_k)_{gen}$ denote the space of functions on $\overline{G}$ which are genuine, locally constant, compactly supported, and bi-B_k-invariant. This is the (metaplectic) *generalized Hecke algebra* on $\overline{G}$.

We will study this algebra in Section 4. Note that it depends on the splitting of the cocycle on B_k. To have a measure on $\overline{G}$ which fits with the measure on G used in §3, we shall also assume that μ_n has measure 1.

2. Some lemmas

Let $\mathbb{F}_q$ denote the residue field of F.

Lemma 3. *For $k > 0$, $B_k/B_{k+1} \cong \mathfrak{b}_k/\mathfrak{b}_{k+1} \cong \mathbb{F}_q^3$, as abelian groups.* (*The second isomorphism is not canonical.*)

Proof. The graph of the the first isomorphism is

$$\{(g\cdot B_{k+1}, X+\mathfrak{b}_{k+1}) \mid (1+X+\mathfrak{b}_{k+1})\cap g\cdot B_{k+1}\neq\emptyset\}.$$

It is also a special case of a general result of Morris (Proposition 3.4 in [M]). The second isomorphism is due to the fact that the map

$$\begin{pmatrix} a & b \\ \pi c & -a \end{pmatrix} \longmapsto \begin{pmatrix} a \pmod{\pi} & b \pmod{\pi} \\ \pi(c \pmod{\pi}) & -a \pmod{\pi} \end{pmatrix}$$

has kernel $\cong \mathbb{F}_q^3$. $\square$

The following result is stated as Proposition 2.2 in Chapter 3 of [H].

Lemma 4. *In the notation of Definition* 1*, let f_h denote the characteristic function of the double coset KhK in H. We have, f_1 is a unit of $\mathcal{H}(H//K)$. More generally, if*

$$\mathrm{vol}(Kh_1K)\mathrm{vol}(Kh_2K) = \mathrm{vol}(Kh_1h_2K)$$

then $f_{h_1} * f_{h_2} = f_{h_1h_2}$.

We need a version of the previous lemma for genuine functions.

For $\overline{x} = (x,\zeta) \in \overline{G}$ and $g\in G$, let

$$\overline{f}_g(\overline{x}) = \begin{cases} \zeta, & x \in B_k g B_k, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\overline{f}_g$ is genuine. Let

$$\phi_1 * \phi_2(g) = \int_{\overline{G}} \phi_1(x)\phi_2(x^{-1}g)dx, \quad g\in\overline{G}.$$

Now we prove a genuine analog of the general Lemma 4 above.

Lemma 5. *If* $\mathrm{vol}(B_k)=1$ *and if*

$$\mathrm{vol}(B_kg_1B_k)\mathrm{vol}(B_kg_2B_k) = \mathrm{vol}(B_kg_1g_2B_k), \qquad \text{for } g_1,g_2\in G,$$

then $\overline{f}_{g_1} * \overline{f}_{g_2} = \overline{f}_{g_1g_2}$.

Proof. We compute

$$\begin{aligned}
\overline{f}_{g_1} * \overline{f}_{g_2}(g,1) &= \int_{\overline{G}} \overline{f}_{g_1}(x)\overline{f}_{g_2}(x^{-1}(g,1))dx \\
&= \int_G \overline{f}_{g_1}(x,1)\overline{f}_{g_2}((x^{-1}g,1))\beta(x,x^{-1}g)^{-1}dx \\
&= \int_G char(\overline{B_k}(g_1,1)\overline{B_k})(x,1)char(\overline{B_k}(g_2,1)\overline{B_k})(x^{-1}g,1)\beta(x,x^{-1}g)^{-1}dx \\
&= \int_G f_{g_1}(x,1)f_{g_2}(x^{-1}g,1)\beta(x,x^{-1}g)^{-1}dx.
\end{aligned}$$

The above equations imply that the support of this integral is contained in the support of $f_{g_1} * f_{g_2}$. By Lemma 4, $f_{g_1} * f_{g_2} = f_{g_1g_2}$, so

$$\mathrm{supp}(\overline{f}_{g_1} * \overline{f}_{g_2}) \subset \mathrm{supp}(\overline{f}_{g_1g_2}).$$

Since the support each element of the Hecke algebra contains at least one double B_k-coset, the above inclusion must be an equality. Therefore, we must have $\overline{f}_{g_1} * \overline{f}_{g_2} =$

$c\overline{f}_{g_1g_2}$ for some constant c. Integrating both sides over G and using the hypothesis implies $c = 1$. □

As in the above proof, we have the following expression for the convolution in terms of the cocycle.

Lemma 6. *We have*

$$\overline{f}_{g_1} * \overline{f}_{g_2}(g,\zeta) = \zeta \int_G \overline{f}_{g_1}(x,1)\overline{f}_{g_2}(x^{-1}g,1)\beta(x,x^{-1}g)^{-1}dx,$$

for any $g_1, g_2 \in G$.

Note that the generalized metaplectic Hecke algebra depends on the cocycle β so let us denote this dependence temporarily by $\mathcal{H}_\beta(\overline{G}//B_k)_{gen}$. If the cocycle is changed to an equivalent one, say to β', where

$$\beta(g,h) = \beta'(g,h)s(g)s(h)s(gh)^{-1}$$

is the cocycle modified by s and $s : G \to \mu_n$ is any function satisfying $s(1) = 1$ (for example, the Kubota splitting [KP]), then the two algebras are isomorphic.

Lemma 7. *Let* β, β' *be as above and let* $s : G \to \mu_n$ *satisfy* $s(1) = 1$. *There is a canonical isomorphism*

$$\phi : \mathcal{H}_\beta(\overline{G}//B_k)_{gen} \cong \mathcal{H}_{\beta'}(\overline{G}//B_k)_{gen},$$

as algebras, defined by sending $f(g,\zeta) = \zeta f(g,1)$ *to* $\phi(f)(g,\zeta) = \zeta s(g)f(g,1)$. *In other words, under this mapping, we have*

$$\phi(f_1 *_{\beta'} f_2)(g,\zeta) = (\phi(f_1) *_\beta \phi(f_2))(g,\zeta),$$

where $*_\beta$ *denotes the convolution with respect to the* β *cocycle*

$$(\phi_1 *_\beta \phi_2)(g,\zeta) = \zeta \int_G \phi_1(x,1)\phi_2((x^{-1}g,1))\beta(x,x^{-1}g)^{-1}dx$$

and, similarly, $*_{\beta'}$ *denotes the convolution with respect to the* β' *cocycle.*

Proof. The verification of this is straightforward. □

3. The structure of $\mathcal{H}(G//B_k)$

We want to determine the structure of the generalized Hecke algebra as a finitely generated algebra with generators and relations. Then we will do the same for the metaplectic analog and compare the two.

The affine Weyl group W_a of G is generated by s_1, s_2 and the Weyl group W is generated by s_1. We will use the Iwahori decomposition

$$G = BW_aB$$

to determine the structure of the generalized Hecke algebra.

Recall $(Ad\,g)(x) = gxg^{-1}$. Let $S = \{s_1, s_2, t_0, t_0^{-1}\} \cup B$.

Proposition 8. *The algebra* $\mathcal{H}(G//B_k)$ *is generated by the functions* f_g, $g \in S$. *Assume that* $\mathrm{vol}(B_k) = 1$. *These elements are subject to the relations below.*

(A) *Relations for the* s_i*'s:*

  (i) $f_{s_1} * f_{s_1} = c\sum_x f_x$, *for* $x \in ((-1)\cdot(Ad\,s_1)(B_k)\cdot B_k)/B_k$, *where* $c = \mathrm{vol}(B_k s_1 B_k)$,

  (ii) $f_{t_0} * f_{s_1} = f_{s_1} * f_{t_0^{-1}}$.

(B) *Relations for elements of* B*:*
  (i) f_1 *is the identity of* $\mathcal{H}(G//B_k)$*,*
  (ii) $f_x * f_y = f_{xy}$*, for* $x, y \in B$*,*
  (iii) $f_{xg} = f_{gx} = f_g$*, for all* $x \in B_k,\ g \in G$*.*

(C) *Mixed relations:*
  (i) $f_{s_i} * f_x = f_{Ad\, s_i(x)} * f_{s_i}$*, for* $x \in B \cap (Ad\, s_i)(B)$*,*
  (ii) $f_{s_1} * f_{u_+(x)} * f_{s_1} = cf_{Ad\, s_1(u_+(x))} = cf_{u_-(-x)}$*, for* $x \in \mathcal{O}^\times$*, where* $c = \mathrm{vol}(B_k s_i B_k)^2$*,*
  (iii) $f_{s_2} * f_{u_+(x)} * f_{s_2} = cf_{Ad\, s_2(u_+(x))} = cf_{u_-(\pi x)}$*, for* $x \in \mathcal{O}^\times$*, where* $c = \mathrm{vol}(B_k s_i B_k)^2$*,*
  (iv) $f_{t_0} * f_g * f_{t_0^{-1}} = cf_{t_0 g t_0^{-1}}$*, for all* $g \in B$*, where*

$$c = \mathrm{vol}(B_k t_0 B_k)^2/\mathrm{vol}(B_k t_0 g t_0^{-1} B_k).$$

(*The relations* C(ii), C(iii) *must be expressed in terms of the generators, as in* [H]*. The relation corresponding to* C(ii) *is*

$$f_{s_1} * f_{u_+(x)} * f_{s_1} = c(f_{u_+(-x^{-1})} * f_{s_1} * f_{h(-x)} * f_{u_+(-x^{-1})}).$$

The relation corresponding to C(iii) *is similar.*)

Furthermore, these relations form a defining set of relations for $\mathcal{H}(G//B_k)$ *as an algebra.*

Remark 1. 1. This proposition has an analog for G replaced by $GL(n)$ [H], Theorem 2.1, Chapter 3.
2. Note that though S is infinite, the set of generators $\{f_g \mid g \in S\}$ is finite (and depends on k), so the algebra is indeed finitely generated.
3. Possibly B(iii) and C(iii) are superfluous relations.
4. Note that s_1 has order 4, not order 2 as the analogous generator in [H] does.
5. The constants are computed as follows:

$$\mathrm{vol}(B_k s_1 B_k) = \mathrm{vol}(B_k s_1 B_k s_1^{-1}) = \mathrm{vol}(B_k s_1 B_k s_1^{-1})/\mathrm{vol}(B_k) = \mathrm{vol}(U_k^- B_k/B_k).$$

For $g \in B$, we have

$$\begin{aligned}\mathrm{vol}(B_k t_0 g t_0^{-1} B_k) &= \mathrm{vol}(B_k t_0 B_k \cdot B_k g B_k \cdot B_k t_0^{-1} B_k) \\ &= \mathrm{vol}(B_k t_0 B_k)\mathrm{vol}(B_k g B_k)\mathrm{vol}(B_k t_0^{-1} B_k) = \mathrm{vol}(B_k t_0 B_k)^2.\end{aligned}$$

Here are some general facts we will use in the proof below.

Lemma 9. (a) B_k *is a normal subgroup of* B*.*
(b) *Let* $U_k^+ = \{u_+(x) \mid x \in \pi^k \mathcal{O}_F\}$ *and* $U_k^- = \{u_-(x) \mid x \in \pi^k \mathcal{O}_F\}$*. We have*

$$B_k \cdot (s_1^{-1} B_k s_1) = (s_1^{-1} B_k s_1) \cdot B_k = U_k^- B_k,$$

and $U_k^- B_k$ *is a group with* B_k *as a normal subgroup.*
(c) *In general, we have*

$$f_{g_1} * f_{g_2}(x) = \int_G f_{g_1}(y) f_{g_2}(y^{-1}x)dy = \mathrm{vol}(B_k g_1 B_k \cap x B_k g_2^{-1} B_k).$$

(d) *In general, we have*

$$\int_G f * g(x)dx = \int_G f(y)dy \int_G g(x)dx,$$

$$\int_G (f * g * h)(x)dx = \int_G f(x)dx \int_G g(y)dy \int_G h(z)dz.$$

Proof. Parts (a) and (b) may be proven by direct matrix calculations. Parts (c), (d) are simple consequences of the definitions. □

Proof of Proposition 8. We shall follow the ideas in [H], proof of Theorem 2.1.

We begin by verifying A(i): $f_{s_1} * f_{s_1} = q\sum_{x\in X} f_x$, where x runs over a complete set X of representatives of $((-1)\cdot(Ad\, s_1)(B_k)\cdot B_k)/B_k$.

As in [H], parts (a), (b) of Lemma 9 imply that

$$f_{s_1} * f_{s_1} = c\cdot \chi_{(-1)\cdot U_k^- B_k},$$

where χ_S denotes the characteristic function of a subset S of G and where $c \neq 0$ is some constant. In other words,

$$f_{s_1} * f_{s_1}(g) = c\sum_{x\in X} f_x(g), \quad g\in G.$$

Plugging $g=1$ into both sides, we obtain

$$\mathrm{vol}(B_k s_1 B_k) = c.$$

For part A(ii), note that part (c) of Lemma 9 immediately reduces the proof of A(ii) to the comparison of two volume integrals. The equality of these two expressions follow from the claim $\mathrm{vol}(B_k t_0^{-1} B_k g \cap B_k s_1 B_k) = \mathrm{vol}(g B_k t_0 B_k \cap B_k s_1 B_k)$. This is not trivial but the hard part is to show that $B_k t_0 B_k s_1 B_k = B_k s_1 B_k t_0^{-1} B_k$. To see this, pick $b_1, b_2, b_3 \in B_k$ and note

$$b_1 t_0 b_2 s_1 b_3 = b_1 t_0 s_1 u_1 b_2' = b_1 s_1 t_0^{-1} u_1 b_2'$$

for some $u_1 \in U_k^-$ and $b_2' \in B_k$ (we've used part (a) of Lemma 9). A matrix calculation shows that this is equal to $b_1 s_1 b_4 t_0^{-1} b_2'$, for some $b_4 \in B_k$. This shows that $B_k t_0 B_k s_1 B_k \subset B_k s_1 B_k t_0^{-1} B_k$. The other inclusion is proven similarly. From this A(ii) follows without too much difficulty.

Part (c) of Lemma 9 implies part B(i) of the proposition.

We now verify part B(ii). We know $\mathrm{vol}(B_k) = 1$ implies $\mathrm{vol}(bB_k) = 1$, for each $b\in B$. This and Lemma 4 implies part B(ii).

Part B(iii) follows from the definition of f_g.

We shall now verify C(i). Since B_k is normal in B, we have

$$\mathrm{vol}(B_k s_i b B_k) = \mathrm{vol}(B_k s_i b B_k b^{-1} b) = \mathrm{vol}(B_k s_i B_k b).$$

Thus $\mathrm{vol}(B_k s_i b B_k) = \mathrm{vol}(B_k s_i B_k b) = \mathrm{vol}(B_k s_i B_k)$. This and Lemma 4 implies $f_{s_i} * f_x = f_{s_i x}$, for $x\in B$. Likewise, if $s_i x s_i^{-1} \in B$ then $f_{s_i x s_i^{-1}} * f_{s_i} = f_{s_i x}$. This implies C(i).

We next verify C(ii). The proof of this part is sketched in [H]. The argument in [H] gives $f_{s_1} * f_{u_+(x)} * f_{s_1} = c f_{u_-(-x)}$, for $x\in\mathcal{O}^\times$, for some constant $c\neq 0$. Here c may be evaluated using part (d) of Lemma 9, which implies

$$c = \frac{\mathrm{vol}(B_k s_i B_k)^2 \mathrm{vol}(B_k u_+(x) B_k)}{\mathrm{vol}(B_k u_-(x) B_k)} = \mathrm{vol}(B_k s_i B_k)^2.$$

(Here we've used the fact that $u_+(x)\in B$, so $\mathrm{vol}(B_k u_+(x) B_k) = \mathrm{vol}(B_k u_+(x)) = \mathrm{vol}(B_k) = 1$, and the fact that, for $k>0$, $s_1^{-1} B_k s_1$ is also a normal subgroup of B, so $\mathrm{vol}(B_k u_-(x) B_k) = \mathrm{vol}(s_1^{-1} B_k s_1 u_+(-x) s_1^{-1} B_k s_1) = \mathrm{vol}(s_1^{-1} B_k s_1 u_+(-x)) = \mathrm{vol}(s_1^{-1} B_k s_1) = 1$.)

The proof of C(iii) is very similar to part (ii), so is omitted.

We shall now verify C(iv). We already know that, as a consequence of Lemma 4, we have $f_{t_0} * f_g = f_{t_0 g}$ for all $g \in B$. This and part (c) of the above lemma implies that

$$f_{t_0} * f_g * f_{t_0^{-1}}(x) = \mathrm{vol}(B_k t_0 g B_k \cap x B_k t_0 B_k).$$

This volume depends on g and x, or more precisely on the left cosets gB_k and xB_k. In other words, we may replace g by any gb, $b \in B_k$, without changing the value of the volume. The above volume is non-zero if and only if $xb_1t_0b_2 = b_3t_0gb_4$, for some b_1, b_2, b_3, b_4 in B_k. By replacing g by a suitable element in gB_k, we may assume $x \in B_k t_0 g t_0^{-1} B_k$. The support of $f_{t_0} * f_g * f_{t_0^{-1}}$ is therefore contained in (and thus must be equal to) the support of $f_{t_0 g t_0^{-1}}$. Therefore, there is a constant c such that $f_{t_0} * f_g * f_{t_0^{-1}} = c \cdot f_{t_0 g t_0^{-1}}$. Part C(iv) follows from part (d) of the above lemma.

It remains to show that this list of relations is a defining set of relations for the algebra. To this end, we slightly modify the proof of [H], Chapter 3, Theorem 2.1, as follows.

For $g \in S$, let $\tilde{f}_g$ denote an abstract element. Let $\tilde{\mathcal{H}}$ denote the free algebra generated by these elements satisfying the relations (A)-(C) in the proposition. We want to show that the obvious map

$$\tilde{\mathcal{H}} \to \mathcal{H}(G//B_k) \tag{1}$$

is an isomorphism.

If $w \in W_a =< s_1, t_0 >$ has the expression $w = s_1^a t_0^b$, where $a \in \{0,1\}$ and b is an integer, then define $\tilde{f}_w = f_{s_1}^a * f_{t_0}^b$. Let $\tilde{\mathcal{J}}$ denote the vector space span generated by the elements $\tilde{f}_g = \tilde{f}_x * \tilde{f}_w * \tilde{f}_y$, where $g = xwy$, for $x, y \in B$ and $w \in W_a$. As in [H], pages 38-39, the relations (B,C) imply that $\tilde{f}_g$ depends only on $B_k g B_k$, not on the particular representation $g = xwy$. We claim that $\tilde{\mathcal{J}}$ is invariant under convolution by an abstract generator $\tilde{f}_g$, $g \in S$. The proof of this invariance property, in [H] pages 38-39, uses a subset of the relations (or ones very similar) given in (A)-(C) above. Therefore, it works in this case almost verbatim and we omit the details. From this invariance property it follows that $\tilde{\mathcal{J}}$ contains the abstract generators. However, this forces $\tilde{\mathcal{J}}$ to contain all of $\tilde{\mathcal{H}}$. We conclude that each element of $\tilde{\mathcal{H}}$ may be written as a sum of elements of the form $\tilde{f}_g = \tilde{f}_x * \tilde{f}_w * \tilde{f}_y$, where $g = xwy$. (One may think of this as a "canonical form" for the generators.) This, using a relatively simple argument involving the definitions (see [H]), implies that the map (1) is injective.

Since B_k is normal in B, we have

$$\mathrm{vol}(B_k b_1 g b_2 B_k) = \mathrm{vol}(b_1 b_1^{-1} B_k b_1 g b_2 B_k b_2^{-1} b_2) = \mathrm{vol}(b_1 B_k g B_k b_2) = \mathrm{vol}(B_k g B_k).$$

This and Lemma 4 implies $f_{b_1} * f_g * f_{b_2} = f_{b_1 g b_2}$, for $g \in W^a$. Combining this with the Bruhat decomposition, we see that $\mathcal{H}(G//B_k)$ is generated by elements of the form $f_{b_1} * f_g * f_{b_2}$, for $g \in W^a$. Therefore, the map (1) is surjective. □

4. The structure of $\mathcal{H}(\overline{G}//B_k)_{gen}$

This section is the metaplectic analog of the previous section. We want to determine the structure of the metaplectic generalized Hecke algebra as a finitely generated algebra with generators and relations.

We shall always assume that k is chosen so large that $\overline{B_k}$ splits. By Lemma 7, we can (and do) choose the cocycle β' defining $\mathcal{H}(\overline{G}//B_k)_{gen} = \mathcal{H}_{\beta'}(\overline{G}//B_k)_{gen}$ in such a way that the splitting of $\overline{B}_k$ is trivial. If the residual characteristic of F is relatively prime to $2n$ then this condition is vacuous since in that case, in fact, $\overline{B}$ splits.

Let the Haar measure on $\overline{G}$ be normalized so that $\overline{B_k}$ has measure one. Convolution gives $\mathcal{H}(\overline{G}//B_k)_{gen}$ the structure of an algebra.

Proposition 10. *Assume that* $\mathrm{vol}(\overline{B}_k) = 1$*, that k is chosen so large that $\overline{B_k}$ splits, and choose the splitting cocycle as above. The algebra $\mathcal{H}(\overline{G}//B_k)_{gen}$ is generated by the functions $\overline{f}_g$, $g \in S$. These elements are subject to the same relations as in Proposition 8, except that f_g must be replaced by $\overline{f}_g$. Furthermore, these relations form a defining set of relations for $\mathcal{H}(\overline{G}//B_k)_{gen}$ as an algebra.*

Proof. First we must verify that the claimed relations do indeed hold.

A(i): By Lemma 6, we have

$$\overline{f}_{s_1} * \overline{f}_{s_1}(\overline{x}) = \int_{B_k s_1 B_k} \overline{f}_{s_1}(y^{-1}x)\beta'(y^{-1},x)\beta'(y,y^{-1})dy.$$

Because of this, for basically the same reason as in the non-metaplectic case, $\overline{f}_{s_1} * \overline{f}_{s_1}$ is supported on $\overline{((-1)\cdot(Ad\, s_1)(B_k)\cdot B_k)}$. The same reasoning as in the non-metaplectic case implies A(i).

A(ii): Both $\overline{f}_{t_0} * \overline{f}_{s_1}$ and $\overline{f}_{s_1} * \overline{f}_{t_0^{-1}}$ are genuine functions which, by the proof of part A(ii) in Proposition 8, have the same support, in a neighborhood of $(t_0 s_1, \pm 1)$. We have, as in the proof of A(ii) in the non-metaplectic case,

$$\overline{f}_{t_0} * \overline{f}_{s_1}(g,1) = \int_{\overline{B_k s_1 B_k}} \overline{f}_{t_0}((g,1)y^{-1})dy = \int_{B_k t_0^{-1} B_k g \cap B_k s_1 B_k} \beta'(y^{-1},g)dy$$

and

$$\overline{f}_{s_1} * \overline{f}_{t_0^{-1}}(g,1) = \int_{g B_k t_0 B_k g \cap B_k s_1 B_k} \beta'(y^{-1},g)dy.$$

We know that the ranges of these two integrals are equal, by the proof in the non-metaplectic case above. The validity of A(ii) follows.

B(i): For example, for $g \in \overline{G_s}$ we have

$$\phi * \overline{f}_1(g) = \int_{\overline{G}} \phi(gx^{-1})\overline{f}_1(x)dx = \int_{\overline{B_k}} \phi(gx^{-1})dx = \phi(g).$$

Similarly, $\overline{f}_1 * \phi = \phi$, so $\overline{f}_1$ is an identity in $\mathcal{H}(\overline{G}//B_k)_{gen}$.

B(ii): We have

$$\overline{f}_x * \overline{f}_y(g,1) = \int_{B_k x B_k} \overline{f}_y(h^{-1}g)\beta'(h^{-1},g)\beta'(h,h^{-1})dh.$$

This is non-zero if and only if $gy^{-1} = xb$ for some $b \in B_k$, since B_k is normal in B. But this implies $\mathrm{supp}(\overline{f}_x * \overline{f}_y) \subset \overline{B_k xy B_k}$. The claimed equality now follows.

B(iii): This is an immediate consequence of Lemma 5.

C(i): The proof of this identity is analogous to the proof in the non-metaplectic case (using Lemma 5 in place of Lemma 4 where appropriate) and is omitted.

C(ii), C(iii), C(iv): These are proven exactly as in the non-metaplectic case so we only sketch the argument. The idea is that the proof in the non-metaplectic case shows that each of these triple convolutions is supported on a single double $\overline{B}_k$-coset. On the other hand, the support of each function in the Hecke algebra contains at least one double $\overline{B}_k$-coset, since different double $\overline{B}_k$-cosets are disjoint. Since convolutions preserve genuineness, those functions whose support is precisely one double $\overline{B}_k$-coset are the constant multiples of the generators. This identity holds on all of $\overline{G}$. The constant multiple can be computed as in the non-metaplectic case and since the measures differ only by the measure on μ_n, the volume computations are similar.

It remains to show that this list of relations is a defining set of relations for the algebra. This, too, is exactly the same as the non-metaplectic case, so is omitted. $\square$

5. Application to representation theory

The following is our main result:

Theorem 11. *If $k > 0$ is such that $\overline{B_k}$ splits then the map $f_g \longmapsto \overline{f}_g$, $g \in S$, defines a bijection of algebras $\mathcal{H}(G//B_k) \to \mathcal{H}(\overline{G}//B_k)_{gen}$.*

Proof. This is an immediate consequence of Propositions 8 and 10. $\square$

Corollary 12. *If $k > 0$ is such that $\overline{B_k}$ splits then the bijection $\mathcal{H}(G//B_k) \to \mathcal{H}(\overline{G}//B_k)_{gen}$ induces a set-theoretic bijection $\mathcal{H}(\overline{G}//B_k)^{\wedge}_{gen} \to \mathcal{H}(G//B_k)^{\wedge}$ on the sets of equivalence classes of irreducible (genuine, in the case of $\overline{G}$) representations. Furthermore, if*

$$\begin{array}{ccc} \mathcal{H}(G//B_k) & \to & \mathcal{H}(\overline{G}//B_k)_{gen} \\ f & \longmapsto & \overline{f} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H}(\overline{G}//B_k)^{\wedge}_{gen} & \to & \mathcal{H}(G//B_k)^{\wedge} \\ \overline{\pi} & \longmapsto & \pi \end{array}$$

then $\mathrm{tr}(\overline{\pi}(\overline{f})) = \mathrm{tr}(\pi(f))$.

Proof. Let $\eta : \mathcal{H}(G//B_k) \to \mathcal{H}(\overline{G}//B_k)_{gen}$ and let

$$\eta^* : \mathcal{H}(\overline{G}//B_k)^{\wedge}_{gen} \to \mathcal{H}(G//B_k)^{\wedge}.$$

Suppose $\overline{\pi}(\overline{f}) = \overline{\pi}'(\overline{f})$, for all $\overline{f} \in \mathcal{H}(\overline{G}//B_k)_{gen}$. If $\overline{f} = \phi(f)$, for some $f \in \mathcal{H}(G//B_k)$, and if $\pi = \eta^*(\overline{\pi})$ and $\pi' = \eta^*(\overline{\pi}')$, for some $\pi, \pi' \in \mathcal{H}(G//B_k)^{\wedge}$, then $\pi(f) = \pi'(f)$, for all $f \in \mathcal{H}(G//B_k)$. This implies $\pi = \pi'$, so η^* is injective.

The equality of the traces is by definition of η. $\square$

Definition 13. Let K denote a compact open subgroup of $\overline{G}$. A locally constant complex-valued function f on $\overline{G}$ is called *left* (resp., *right*) *Hecke finite* if the vector space spanned by all functions of the form $h * f$ (resp., $f * h$), for $h \in \mathcal{H}(\overline{G}//K)$, is finite dimensional. Let $\mathcal{A}_{gen}(\overline{G})$ denote the space of all locally constant complex-valued genuine functions f on $\overline{G}$ for which

- f is double K-finite,
- f is right- and left-Hecke finite.

This space is called the space of *matrix coefficients* of $\overline{G}$. If (π, V) is a genuine admissible representation of $\overline{G}$ then the span of the elements $f(g) = \langle v_0^*, \pi(g)v_0\rangle$ $(g \in \overline{G})$, for all $v_0 \in V$ and $v_0^* \in V^*$, the contragredient of V, is called the space of *matrix coefficients of* π and is denoted $\mathcal{A}_{gen}(\pi)$.

Let ρ (resp, λ) denote the right (resp., left) regular representation of $\overline{G}$ on $\mathcal{A}_{gen}(\overline{G})$. The next proposition establishes that the metaplectic analogs of (the relevant parts of) Corollary 1.10.6 in [Sil] for $\overline{G}$ are true. The result below implies that we can "explicitly" realize the space of any genuine irreducible representation of $\overline{G}$ as a subspace of $\mathcal{A}_{gen}(\overline{G})$, a fact we will make use of later.

Proposition 14. *Let $f \in \mathcal{A}_{gen}(\overline{G})$ and let V_f denote the $\overline{G}$-module spanned by the translates $\rho(g)f$, $g \in \overline{G}$, and let π denote the restriction of ρ to V_f. We have*

(a) *There exist elements $v_0 \in V_f$ and $v_0^* \in V_f^*$, the contragredient of V_f, such that $f(g) = \langle v_0^*, \pi(g)v_0\rangle$, for all $g \in \overline{G}$.*
(b) *$\mathcal{A}_{gen}(\pi)$ is spanned by $\lambda(x)\rho(y)f$, $x, y \in \overline{G}$.*

Proof. The proof of this follows from the general arguments given in [Sil]. The results in that part of the book [Sil] hold for more general totally disconnected groups, so one need only check that the argument remains valid when "genuineness" is also assumed. □

We need to recall some facts regarding the relationship between smooth representations of $\overline{G}$ and finite dimensional $\mathcal{H}(\overline{G}//B_k)_{gen}$-modules. To this end, we recall briefly how one can construct smooth representations of $\overline{G}$ from finite dimensional $\mathcal{H}(\overline{G}//B_k)_{gen}$-modules - both the so-called "induced" and "produced" modules in §2 of Borel [Bo] will do this job. The "inverse" construction is then given along with some of the basic properties of these constructions in Proposition 16 below.

Definition 15. If (r, W) is a finite dimensional $\mathcal{H}(\overline{G}//B_k)$-module then we define $(I(r), I(W))$ to be the *induced* $\overline{G}$-module and $(P(r), P(W))$ to be the (smooth) *produced* $\overline{G}$-module constructed in [Bo]. Analogously, if (r, W) is a finite dimensional $\mathcal{H}(\overline{G}//B_k)_{gen}$-module then we define $(I_{gen}(r), I_{gen}(W))$ to be the *induced* genuine $\overline{G}$-module and $(P_{gen}(r), P_{gen}(W))$ to be the *produced* genuine $\overline{G}$-module obtained by replacing $C_c(\overline{G}//B_k)$ by $C_c(\overline{G}//B_k)_{gen}$ in the contructions in [Bo].

Proposition 16. *Let (r, W) be a finite dimensional $\mathcal{H}(\overline{G}//B_k)_{gen}$-module and let (π, V) be a genuine smooth representation of $\overline{G}$.*

(a) *There are natural $\mathcal{H}(\overline{G}//B_k)_{gen}$-module isomorphisms*

$$W \cong I_{gen}(W)^{B_k} \quad \text{and} \quad W \cong P_{gen}(W)^{B_k}.$$

(b) *$I_{gen}(W)$ is generated, as a $\overline{G}$-module, by $I_{gen}(W)^{B_k}$.*
(c) *The natural restriction maps yield isomorphisms*

$$\rho_P : \mathrm{Hom}_{\overline{G}}(V, P_{gen}(W)) \to \mathrm{Hom}_{\mathcal{H}(\overline{G}//B_k)_{gen}}(V^{B_k}, W).$$

and

$$\rho_I : \mathrm{Hom}_{\overline{G}}(I_{gen}(W), V) \to \mathrm{Hom}_{\mathcal{H}(\overline{G}//B_k)_{gen}}(W, V^{B_k}).$$

(d) *(r,W) is an irreducible $\mathcal{H}(\overline{G}//B_k)_{gen}$-module if and only if $(I_{gen}(r), I_{gen}(W))$ is an irreducible representation of $\overline{G}$. Moreover, if (r,W) is an irreducible $\mathcal{H}(\overline{G}//B_k)_{gen}$-module then $I_{gen}(r) \cong P_{gen}(r)$.*

Proof. In the case $n = 1$, the statements in the proposition above are either in §§1-2 [Bo] as stated or follow from Borel's results using Schur's lemma (see for example, Lemma 3.14 in [CR]). In general, one need only check that the argument remains valid when "genuineness" is also assumed. $\square$

Definition 17. We define a genuine admissible irreducible representation of $\overline{G}$ to be *supercuspidal* if and only if its matrix coefficients are compactly supported. We define a genuine admissible irreducible representation of $\overline{G}$ to be *square-integrable* if and only if its matrix coefficients belong to $L^2(\overline{G})$.

We define an irreducible finite dimensional $\mathcal{H}(\overline{G}//B_k)_{gen}$-module (r,W) to be *supercuspidal* (resp., *square-integrable*) if and only if $I_{gen}(r)$ is a supercuspidal (resp., square-integrable) representation of $\overline{G}$.

Next we observe that, as in the case of $GL(n)$, the correspondence of Corollary 12 preserves supercuspidals and discrete series representations ([H], page 33).

Corollary 18. *Assume $k > 0$ is as in Corollary 12.*

- *An admissible genuine representation $\overline{\pi} \in \mathcal{H}(\overline{G}//B_k)^{\wedge}_{gen}$ is supercuspidal if and only if π is, where $\pi, \overline{\pi}$ are as in Corollary 12.*
- *$\overline{\pi}$ is square-integrable if and only if π is, where $\pi, \overline{\pi}$ are as in Corollary 12.*

Proof. Suppose that the mapping $\mathcal{H}(\overline{G}//B_k)^{\wedge}_{gen} \to \mathcal{H}(G//B_k)^{\wedge}$ sends a representation $(\overline{r}, \overline{W})$ to a representation (r,W). Note that the construction of correspondence in Corollary 12 implies that $W = \overline{W}$. From the construction of $I_{gen}(\overline{W}) = C_c(\overline{G}/B_k)_{gen} \otimes_{\mathcal{H}(\overline{G}//B_k)_{gen}} \overline{W}$ (more precisely from the definition of $\otimes_{\mathcal{H}(\overline{G}//B_k)_{gen}}$), we see that the $r(\overline{f}_g)$-action on $\overline{W} \cong I_{gen}(\overline{W})^{B_k}$ corresponds to the $\overline{f}_g$-action (via convolution) on $C_c(\overline{G}/B_k)_{gen}$. Identifying $\overline{W}$ with $I_{gen}(\overline{W})^{B_k}$, we see that each element of $\overline{W}$ is Hecke-finite and may therefore be regarded as a matrix coefficient of $\overline{G}$. Let $(\overline{\pi}, \overline{V})$ denote the $\overline{G}$-module generated by an element $w \in \overline{W} \cong I_{gen}(\overline{W})^{B_k}$. By Proposition 14 (or Proposition 16 (b)), $\overline{\pi} = I_{gen}(\overline{r})$. From this and the fact that $W = \overline{W}$, it follows that if the elements of $\overline{W}$ are compactly supported (when regarded as matrix coefficients of $\overline{G}$) then the elements of W are also are compactly supported (when regarded as matrix coefficients of G). Now assume that $(\overline{r}, \overline{W})$ is supercuspidal, so the matrix coefficients of $I_{gen}(\overline{r})$ have compact support. As indicated above, (r,W) is supercuspidal as well. This established the first part of the corollary.

The second part of the corollary is proved similarly, hence is omitted. $\square$

6. The Iwahori algebra

In the case where $k = 0$ and $\gcd(p, 2n) = 1$, our result is a special case of a general isomorphism between Iwahori algebras, due to Savin [S], which we recall:

Theorem 19. (Savin) *Assume $\gcd(p, 2n) = 1$. There is an isomorphism of algebras $\mathcal{H}(\overline{G}//B)_{gen} \to \mathcal{H}(G//B)$.*

Corollary 20. *Assume* $\gcd(p,2n)=1$. *The isomorphism*

$$\mathcal{H}(\overline{G}//B)_{gen} \to \mathcal{H}(G//B)$$

induces a set-theoretic bijection $\mathcal{H}(\overline{G}//B)^{\wedge}_{gen} \to \mathcal{H}(G//B)^{\wedge}$ *on the sets of equivalence classes of irreducible (genuine, in the case of* $\overline{G}$*) representations with a* B*-fixed vector. Furthermore, if*

$$\begin{array}{ccc} \mathcal{H}(G//B) & \to & \mathcal{H}(\overline{G}//B)_{gen} \\ f & \longmapsto & \overline{f} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H}(\overline{G}//B)^{\wedge}_{gen} & \to & \mathcal{H}(G//B)^{\wedge} \\ \overline{\pi} & \longmapsto & \pi \end{array}$$

then $\mathrm{tr}(\overline{\pi}(\overline{f})) = \mathrm{tr}(\pi(f))$.

References

[Bo] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976) 233–259, MR 56 #3196.

[CR] C. Curtis, I. Reiner, *Methods of representation theory*, I, Wiley Interscience, John Wiley and Sons, New York, 1981, MR 82i:20001.

[F] Y. Flicker, *Automorphic forms on covering groups of* $GL(2)$, Invent. Math. **57** (1980) 119–182, MR 81m:10057.

[FK] Y. Flicker and D. Kazhdan, *Metaplectic correspondence*, Publ. Math. IHES **64** (1986) 53–110, MR 88d:11049.

[G] S. Gelbart, *Weil's representation and the spectrum of the metaplectic group*, Lect. Notes in Math., No. 530, Springer-Verlag, Berlin, 1976, MR 54 #12654.

[H] R. Howe (with the collaboration of A. Moy), *Harish-Chandra homomorphisms for p-adic groups*, CBMS Regional Conference Series in Mathematics, No. 59, Amer. Math. Soc., Providence, 1985, MR 87h:22023.

[KP] D. Kazhdan and S. Patterson, *Metaplectic forms*, Publ. Math. IHES **59** (1984) 35–142, MR 85g:22033.

[K] T. Kubota, *Topological covering of* $SL(2)$ *over a local field*, J. Math. Soc. Japan **19** (1967) 114–121, MR 34 #4264.

[M] L. Morris, *P-cuspidal representations*, Proc. London Math. Soc. **57** (1988) 329–356, MR 89j:22038.

[RS] S. Rallis and G. Schiffmann, *Représentations supercuspidales du groupe métaplectique,* J. Math. Kyoto Univ. **17** (1977) 567–603, MR 58 #16523.

[S] G. Savin, *Local Shimura correspondence*, Math Ann. **280** (1988) 185–190, MR 89h:22018.

[Sch] J. Schultz, *Lifting of characters of* $\overline{SL_2(F)}$ *and* $SO_{2,1}(F)$, *for F nonarchimedean*, PhD thesis (supervised by J. Adams), Univ. Maryland at College Park, expected summer 1998.

[Sil] A. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*, Princeton Univ. Press, Princeton, 1979, MR 81m:22025.

[W] J.-L. Waldspurger, *Correspondence de Shimura*, J. Math. Pures Appl. **59** (1980) 1–132, MR 83f:10029.

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