New York Journal of Mathematics

New York J. Math. **4** (1998) 185–221.

3-Primary v_1-Periodic Homotopy Groups of E_7

Donald M. Davis

Abstract. In this paper we compute the 3-primary v_1-periodic homotopy groups of the exceptional Lie group E_7. This represents the next stage in the author's goal of calculating the v_1-periodic homotopy groups of all compact simple Lie groups (at least when localized at an odd prime). Most of the work goes into calculating the unstable Novikov spectral sequence of $\Omega E_7/Sp(2)$. Showing that this spectral sequence converges to the v_1-periodic homotopy groups in this case utilizes recent results of Bousfield and Bendersky-Thompson.

Contents

1. Introduction

In this paper we compute the 3-primary v_1-periodic homotopy groups of the exceptional Lie group E_7.

The p-primary v_1-periodic homotopy groups of a space X, denoted $v_1^{-1}\pi_*(X;p)$ or just $v_1^{-1}\pi_*(X)$, were defined in [21]. They are a localization of the actual homotopy groups, telling roughly the portion which is detected by K-theory and its operations. If X is a compact Lie group, each $v_1^{-1}\pi_i(X;p)$ is a direct summand of some actual homotopy group of X, and so summands of v_1-periodic homotopy groups of X give lower bounds for the p-exponent of X.

Received May 4, 1998.

Mathematics Subject Classification. 55T15.

Key words and phrases. v_1-periodic homotopy groups, exceptional Lie groups, unstable Novikov spectral sequence.

ISSN 1076-9803/98

After the author computed $v_1^{-1}\pi_*(SU(n);p)$ for odd p in 1989, Mimura proposed the goal of calculating $v_1^{-1}\pi_*(X;p)$ for all compact simple Lie groups X. This has now been achieved in the following cases (X,p):

- X a classical group and p odd ([19]);
- X an exceptional Lie group with $H_*(X;\mathbf{Z})$ p-torsion-free ([13]);
- ($SU(n)$ or $Sp(n)$, 2) ([11],[12]);
- $(G_2,2)$ ([22]), (F_4 or E_6, 3) ([10]), and $(E_7,3)$ (the current paper).

The only cases remaining then are $(E_8, 2$ or 3 or 5) and ($SO(n)$ or F_4 or E_6 or $E_7, 2)$. Several of these appear tractable.

Now we state our main theorem. We usually abbreviate $v_1^{-1}\pi_*(X;3)$ as $v_*(X)$, and denote by $\nu(n)$ the exponent of 3 in the integer n.

Theorem 1.1. *If j is even, then $v_{2j}(E_7)=v_{2j-1}(E_7)=0$. If j is odd, then*

$$v_{2j}(E_7)\approx v_{2j-1}(E_7)\approx\begin{cases}\mathbf{Z}/3\oplus\mathbf{Z}/3^{\min(10,\nu(j-9-2\cdot 3^5)+4)} & \text{if } j\equiv 0 \text{ mod } 3\\ \mathbf{Z}/3\oplus\mathbf{Z}/3^{\min(8,\nu(j-43)+5)} & \text{if } j\equiv 1,7 \text{ mod } 9\\ \mathbf{Z}/3\oplus\mathbf{Z}/3^{\min(14,\nu(j-13-4\cdot 3^8)+5)} & \text{if } j\equiv 4 \text{ mod } 9\\ \mathbf{Z}/9\oplus\mathbf{Z}/3^{\min(19,\nu(j-17-2\delta\cdot 3^{13})+4)} & \text{if } j\equiv 5,8 \text{ mod } 9,\end{cases}$$

where δ equals one of the numbers 2, 5, or 8. If j is odd and $j\equiv 2$ mod 9, then $v_{2j-1}(E_7)\approx\mathbf{Z}/9\oplus\mathbf{Z}/3^{\min(13,\nu(j-11)+4)}$, while

$$v_{2j}(E_7)\approx\begin{cases}\mathbf{Z}/3^3\oplus\mathbf{Z}/3^{\nu(j-11)+3} & \text{if } \nu(j-11)<10\\ \mathbf{Z}/3^3\oplus\mathbf{Z}/3^{12} \text{ or } \mathbf{Z}/3^4\oplus\mathbf{Z}/3^{11} & \text{if } \nu(j-11)\geq 10.\end{cases}$$

An immediate corollary of this work is a lower bound for the 3-exponent of E_7. Recall that the p-exponent of a space X, denoted $\exp_p(X)$, is the largest e such that $\pi_*(X)$ has an element of order p^e. We obtain

Corollary 1.2. *The 3-exponent of E_7 satisfies $\exp_3(E_7)\geq 19$.*

Proof. If X is a compact Lie group, then $v_i(X)\approx \operatorname{dirlim}_{k,m}\pi_{i+4k\cdot 3^m}(X)$. Hence an element of order 3^{19} in $v_{2j}(E_7)$ when $j\equiv 17+2\delta\cdot 3^{13}$ mod $2\cdot 3^{15}$ corresponds to an element of order 3^{19} in some $\pi_n(E_7)$. □

By comparison, the result that we have obtained at other primes is ([13])

$$\exp_p(E_7)\begin{cases}=17 & \text{if } p>17\\ =18 & \text{if } p=17\\ \geq 17 & \text{if } p=7,\ 11, \text{ or } 13\\ \geq 18 & \text{if } p=5.\end{cases}$$

This should be contrasted with the situation for spheres, where $\exp_p(S^{2n+1})=n$ for all odd primes p and all positive integers n by [25] and [18].

Note that in Theorem 1.1, we determine the precise abelian group structure of all groups (with isolated exceptions), whereas in some earlier papers, such as [19], [13], and [10], we had been unable to determine the group structure of most groups $v_{2j-1}(X)$. Because of the insights of [23], we are able not only to resolve the extension questions (group structure) in almost all cases occurring here, but also in those of [13] and [10]. These new results about group structure are presented in Section 3.

Most of the work is calculation of the v_1-periodic unstable Novikov spectral sequence (UNSS) of the space $Y_7 := \Omega E_7/Sp(2)$. The main input is the detailed structure of $H_*(\Omega E_7; \mathbf{Z}/3)$ given in [26] and restated here in Proposition 5.2. The advantage of Y_7 over ΩE_7 is that $BP_*(Y_7)$ is a free commutative algebra, which makes its UNSS easier to calculate. Perhaps the most novel feature of the calculations here is the use of coassociativity to give detailed formulas for BP_*-coaction. The terms which arise in this way play crucial roles in the calculations. The calculations of $v_*(F_4)$ in [10] are essential in the transition from $v_*(Y_7)$ to $v_*(\Omega E_7)$.

Another delicate point is convergence of the v_1-periodic UNSS for Y_7. In Section 4, we use deep recent work of Bousfield and Bendersky-Thompson to prove that the v_1-periodic UNSS converges to $v_*(-)$ for E_7/F_4, which we will show implies similar convergence for Y_7.

The author would like to thank Mamoru Mimura, Pete Bousfield, and especially Martin Bendersky for useful suggestions.

Note added in proof (Oct. 5, 1998): Using a completely different method, based on [17, 9.2], the author has obtained an independent verification of the results for $v_{2j}(E_7)$ given in Theorem 1.1, and shown that in the cases unresolved in Theorem 1.1, $\delta = 5$ and, if j is odd and $\nu(j-11) \geq 10$, then $v_{2j}(E_7) \approx \mathbf{Z}/3^3 \oplus \mathbf{Z}/3^{12}$. This method relies very heavily on computer calculations, and will be discussed in a forthcoming paper.

2. Background in v_1-periodic homotopy and the UNSS

In this section, we review known results, and establish one new useful result about computing the UNSS. Although some of these results are also true when $p = 2$, it will simplify exposition to assume that p is an odd prime.

The v_1-periodic homotopy groups of any topological space X are defined by

$$v_1^{-1}\pi_i(X) = \varinjlim [M^{i+1+kqp^e}(p^e), X], \tag{2.1}$$

where $q = 2p-2$, a notation that will be used consistently throughout this paper, and $M^t(n)$ denotes the Moore space $S^{t-1} \cup_n e^t$. Here the direct limit is taken over increasing values of e and k using Adams maps $M^{t+qp^{e-1}}(p^e) \to M^t(p^e)$ and canonical maps $M^t(p^{e+1}) \to M^t(p^e)$. This definition was given in [21], where their relationship with actual homotopy groups of many spaces was established.

A space X is said to *have an H-space exponent* at the prime p if, for some e and L, $p^e : \Omega^L X \to \Omega^L X$ is null homotopic. It was shown in [21, 1.9] that if X has an H-space exponent, then

$$v_1^{-1}\pi_i(X) \approx \varinjlim \pi_{i+kqp^e}(X),$$

and hence $v_1^{-1}\pi_i(X)$ is a direct summand of some group $\pi_{i+kqp^e}(X)$. To make this final deduction, we need to know that the limit group is a finitely generated abelian group, but this will be the case.

Next we discuss the unstable cobar complex, which can be used to compute the UNSS for many spaces. We will modify and generalize previous treatments of this topic. Let BP be the Brown-Peterson spectrum corresponding to the prime p. Then

$$BP_* = \pi_*(BP) \approx Z_{(p)}[v_1, v_2, \dots],$$

where v_i are the Hazewinkel generators of BP_*. Let

$$\Gamma = BP_*(BP) \approx BP_*[h_1, h_2, \dots],$$

where h_i are conjugates of Quillen's generators t_i. We have $|v_i| = |h_i| = 2(p^i - 1)$. Let $\eta = \eta_R : BP_* \to BP_*(BP)$ be the right unit. We write $h_i v_j$ interchangeably with $\eta(v_j)h_i$; this is the right action of BP_* on Γ.

Let M be a Γ-comodule with coaction map $\psi_M : M \to \Gamma \otimes M$. Tensor products are always over BP_*. The stable cobar complex $SC^*(M)$ is defined by

$$SC^s(M) = \Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma \otimes M,$$

with s copies of Γ, and differential $d : SC^s(M) \to SC^{s+1}(M)$ given by

$$\begin{aligned} (2.2) \quad d(\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m) \;\; &= \;\; 1 \otimes \gamma_1 \otimes \cdots \gamma_s \otimes m \\ &\quad + \sum_{j=1}^{s} (-1)^j \gamma_1 \otimes \cdots \otimes \psi(\gamma_j) \otimes \cdots \otimes \gamma_s \otimes m \\ (2.3) \qquad\qquad\qquad\qquad &\quad +(-1)^{s+1} \gamma_1 \otimes \cdots \otimes \gamma_s \otimes \psi_M(m). \end{aligned}$$

Our unstable cobar complex $VC^*(M)$ is a subcomplex of $SC^*(M)$, consisting of terms satisfying an unstable condition, introduced in the following definition.

Definition 2.4. [6, 3.3] *If M is a nonnegatively graded free left BP_*-module, then $V(M)$ is defined to be the BP_*-span of*

$$\{h^I \otimes m \;\;:\;\; 2(i_1 + i_2 + \cdots) \le |m|\} \subset \Gamma \otimes M,$$

where $I = (i_1, i_2, \dots)$ and $h^I = h_1^{i_1} h_2^{i_2} \cdots$.

This unstable condition will pervade our computations. Note that for odd dimensional classes, this agrees with the module $U(M)$ which has been used most frequently in earlier work of the author and Bendersky. However, it also agrees with the $V(M)$ construction employed in [12] on even-dimensional classes. The novelty here is that it will be applied to a module having classes of both parities.

Define $VC^0(M) = M$, and $VC^s(M) = V(VC^{s-1}(M))$. If M is a Γ-comodule, then the differential d of the stable cobar complex of M induces a differential on the subcomplex $VC^*(M)$. We will usually replace it by the chain-equivalent reduced complex obtained by replacing $V(M)$ by $\ker(V(M) \xrightarrow{\epsilon} M)$ ([5, 2.16]). This has the effect of only looking at terms which have positive grading in each position. The homology groups of this unstable cobar complex are denoted by $\mathrm{Ext}_{\mathcal{V}}^{s,t}(M)$. As observed in [6], these are the usual Ext groups in the abelian category $\mathcal{V}$ of Γ-comodules satisfying the unstable condition in Definition 2.4. Note there is a shift isomorphism

$$(2.5) \qquad \mathrm{Ext}_{\mathcal{V}}^{s,t-1}(BP_* S^{2n}) \approx \mathrm{Ext}_{\mathcal{V}}^{s,t}(BP_* S^{2n+1}),$$

induced by a shift isomorphism of the unstable cobar complexes.

The following generalization of [8, §7] will be very useful to us. Its proof follows some suggestions of Martin Bendersky.

Theorem 2.6. *If X is a simply-connected CW-space, there is a spectral sequence $\{E_r^{s,t}(X), d_r\}$ which converges to the homotopy groups of X localized at p. If X is an H-space, and $BP_*(X)$ is a free commutative algebra, then*

$$E_2^{s,t}(X) = \mathrm{Ext}_{\mathcal{V}}^{s,t}(Q(BP_* X)),$$

where $Q(BP_*X)$ denotes the indecomposable quotient of BP_*X.

This is the UNSS for the space X. We will write $VC^*(X)$ for the complex $VC^*(Q(BP_*X))$, whose homology is $E_2(X)$. We denote by Fr the free commutative algebra functor. If N is a free BP_*-module with basis $B = B_{\mathrm{ev}} \cup B_{\mathrm{od}}$, then $\mathrm{Fr}(N)$ is the tensor product of a polynomial algebra over BP_* on B_{ev} with an exterior algebra on B_{od}.

Proof. The spectral sequence was described in [8]. The determination of E_2 when $BP_*(X)$ is a free commutative algebra is quite similar to that of [9, 6.1] and to the argument on [12, p.346]. Let $M = Q(BP_*X)$, a free BP_*-module.

Let $\mathcal{G}$ denote the category of unstable Γ-coalgebras, and $G(-)$ the associated functor considered in [8]. If N is a free BP_*-module, then $G(N)$ is defined to be $BP_*(BP(N))$, where $BP(N)$ is the 0th space of the Ω-spectrum representing the homology theory $BP_*(-) \otimes N$. If N has basis B, then

$$G(N) \approx BP_*(\prod_{b\in B} \mathbf{BP}_{|b|}) \approx \mathrm{Fr}(\langle h^I b : b \in B,\ 2|I| \le |b|\rangle). \tag{2.7}$$

Here h^I is as in Definition 2.4 with $|I| = \sum i_j$, while $|b|$ denotes the degree of the basis element b. Also $\mathbf{BP}_n$ denotes the nth space in the Ω-spectrum for BP. The first isomorphism in (2.7) is immediate from the definition of G given in [8, 6.3,6.7]. The second isomorphism follows from [33, p.51], which says that $H_*(\mathbf{BP}_n)$ is a polynomial algebra if n is even, and an exterior algebra if n is odd, [31, 4.9], which says that the same thing is then true of $BP_*(\mathbf{BP}_n)$, and [7, p.1040], which interprets conveniently the description of the indecomposables first given in [31]. Note that there is an isomorphism of BP_*-modules

$$Q(G(N)) \approx V(N). \tag{2.8}$$

We claim that

$$BP_*X \xrightarrow{\xi} G(M) \xrightarrow{\to} G(V(M)) \underset{\to}{\xrightarrow{\to}} G(V^2(M)) \cdots \tag{2.9}$$

is an augmented cosimplicial resolution in $\mathcal{G}$. Here the augmentation ξ is the composite

$$BP_*X \xrightarrow{\eta_X} G(BP_*X) \xrightarrow{G(\rho)} G(QBP_*X),$$

where the second morphism applies G to the quotient morphism ρ. The cofaces are of two types:

- $G(V^qM) \xrightarrow{\eta_{G(V^qM)}} G(G(V^qM)) \xrightarrow{G(\rho)} G(V^{q+1}M)$, where $\rho : G(-) \to QG(-) = V(-)$ is the quotient morphism.
- $G(V^i(\psi_{V^{q-i}M}))$, $0 \le i \le q$, where $\psi_N : N \to V(N)$ stabilizes to the Γ-coaction.

The degeneracies $G(V^qM) \to G(V^{q-1}M)$ just do the counit ϵ on one of the V-factors. It is clear that all of these morphisms are in $\mathcal{G}$, and the cosimplicial identities are satisfied as usual. The argument of [12, 3.13] implies that the first type of coface map and the augmentation ξ are algebra morphisms. The second type of coface map is an algebra morphism since it is $BP_*(f)$ for an infinite loop map f, namely the map $BP(N) \xrightarrow{BP(g)} BP(N')$ induced by a BP_*-morphism $N \xrightarrow{g} N'$.

The exactness of the resulting augmented cochain complex

$$0 \to BP_*X \xrightarrow{\xi} G(M) \to G(V(M)) \to G(V^2(M)) \to \cdots \tag{2.10}$$

(obtained using the alternating sum of cofaces as boundaries) follows as in [9, p.387], but we provide details for completeness. (In comparing with [9], it is useful to note that $V(N) \approx \sigma^{-1}U(\sigma N)$.) Since the coface operators are algebra homomorphisms, their alternating sum preserves the filtration of this augmented complex by powers of the augmentation ideal. Let E_0 denote the quotients of the filtration. Then, using (2.8), we have

$$E_0(G(V^q(M))) \approx \mathrm{Fr}(Q(G(V^q(M)))) \approx \mathrm{Fr}(V^{q+1}(M)),$$

and $E_0(BP_*X) \approx \mathrm{Fr}(M)$. Thus $E_0(2.10)$ is the free commutative algebra on the complex

$$0 \to M \to V(M) \to V^2(M) \to \cdots$$

with morphisms the alternating sum of ψ on each V and ψ_M, which is exact by [16, 7.8]. Since the free commutative algebra functor applied to an exact sequence yields an exact sequence, we deduce that (2.10) is exact, and hence yields a resolution in $\mathcal{G}$ of BP_*X.

Hence $\mathrm{Ext}_{\mathcal{G}}(BP_*, BP_*X)$ is equal to the cohomology of the complex obtained by applying $\mathrm{Hom}_{\mathcal{G}}(BP_*, -)$ to the portion of (2.10) after ξ. Since $\mathrm{Hom}_{\mathcal{G}}(BP_*, G(N)) \approx N$, we obtain that $\mathrm{Ext}_{\mathcal{G}}(BP_*, BP_*X)$ is the homology of the complex

$$M \to V(M) \to V^2(M) \to \cdots, \tag{2.11}$$

with differentials as in (2.3). The claim of the theorem follows now from [8, 6.17], which states that $E_2(X) \approx \mathrm{Ext}_{\mathcal{G}}(BP_*, BP_*X)$, and the observation that (2.11) is just our unstable cobar complex, whose homology is $\mathrm{Ext}_{\mathcal{V}}(M)$. □

The following definition will be extremely important.

Definition 2.12. *The excess* $\mathrm{exc}(\gamma)$ *of an element* γ *of* $\bar{\Gamma}^s$ *is defined to be the smallest* n *such that* $\gamma\iota_{2n+1}$ *is an element of* $VC^s(S^{2n+1})$.

This means that if $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_s$, then for $1 \le i \le s$,

$$\gamma_i \otimes (\gamma_{i+1} \cdots \gamma_s \iota_{2n+1})$$

must satisfy 2.4. The following result, which was proved as [20, 4.2], gives a formula for the excess of certain monomials when $s = 2$.

Lemma 2.13. *If* $a \le b$ *and* $a \le d$, *then*

$$\mathrm{exc}(p^a h^b \otimes v^c h^d v^e) = \max\Big(b-(p-1)(c+d), d\Big) - \min\Big(a, |b-(p-1)c-pd|\Big) - (p-1)e.$$

In [4], the v_1-periodic UNSS was defined and shown to satisfy the following very nice property.

Theorem 2.14. *If* p *is odd and* X *is spherically resolved, the* v_1*-periodic UNSS of* X *satisfies*

- $v_1^{-1}E_\infty^{s,t}(X) = v_1^{-1}E_2^{s,t}(X)$, *and is* 0 *unless* $s = 1$ *or* 2 *and* t *is odd.*
- $v_1^{-1}E_\infty^{s,t}(X) \approx v_1^{-1}\pi_{t-s}(X)$ *if* $s = 1$ *or* 2 *and* t *is odd.*
- $v_1^{-1}E_2^{s,t}(X) = \mathrm{dirlim}\, E_2^{s,t+kqp^e}(X)$, *where* e *is chosen sufficiently large, and the direct limit is taken over increasing values of* k *under multiplication by* $v_1^{p^e}$.

Here we say that X is spherically resolved if it can be built from a finite number of odd-dimensional spheres by fibrations. In Section 4, we will show that Theorem 2.14 holds in a certain case in which we cannot prove that X is spherically resolved.

We will use the unstable cobar complex for the unlocalized UNSS, but, as we are dealing exclusively with v_1-periodic classes, we can, in effect, act as if it satisfies the first two properties of Theorem 2.14.

We will make frequent use of the following result for the spheres, which was proved in [20], following [8] and [3]. We introduce here terminology $x \equiv y$ mod S^{2n-1} to mean that $x-y$ desuspends to (or is defined on) S^{2n-1}. For elements of $E_2^s(S^{2n+1})$, we frequently abbreviate $x\iota_{2n+1}$ as x.

Theorem 2.15. 1. *The only nonzero groups* $v_1^{-1}E_2^{s,t}(S^{2n+1})$ *are*

$$v_1^{-1}E_2^{s,2n+1+qm}(S^{2n+1}) \approx \mathbf{Z}/p^e$$

with $s=1$ *or* 2 *and* $e=\min(n,\nu(m)+1)$.

2. *The generator of* $E_2^{1,2n+1+qm}(S^{2n+1})$ *is* $\alpha_{m/e} := d(v_1^m)/p^e$ *and satisfies*

$$\alpha_{m/e} \equiv -v_1^{m-e}h_1^e \mod S^{2e-1}, \tag{2.16}$$

and, if $m = sp^{e-1}$ *with* $s \not\equiv 0$ *mod* p, *and* $e>n$, *then*

$$\alpha_{m/e} \equiv -sv_1^{m-1}h_1 \mod p. \tag{2.17}$$

3. *If* $n \le \nu(m)+1$ *and* $1 \le j \le n$, *then* $d(p^{m-\nu(m)-1-j}h_1^m)\iota_{2n+1}$ *has order* p^j *in* $E_2^{2,2n+1+qm}(S^{2n+1})$. *It equals* $v_1^{m-j-1}h_1 \otimes h_1^j$ *mod* S^{2j-1}.
4. *If* $\nu(m)+1 \le n$ *and* $1 \le j \le \nu(m)+1$, *then* $d(p^{m-n-j}h_1^m)\iota_{2n+1}$ *has order* p^j *in* $E_2^{2,2n+1+qm}(S^{2n+1})$. *It equals* $v_1^{m-n-j+\nu(m)}h_1 \otimes h_1^{n+j-\nu(m)-1}$ *mod* $S^{2n+2j-2\nu(m)-3}$.
5. *The homomorphism* $\Sigma^2 : E_2^{2,2n-1+qm}(S^{2n-1}) \to E_2^{2,2n+1+qm}(S^{2n+1})$ *is injective if* $n \le \nu(m)+1$ *and is multiplication by* p *otherwise.*

Other more technical results proved in earlier works are as follows. Here we begin the practice, which will be continued throughout the paper, of often abbreviating h_1 as h, and v_1 as v. Also, we introduce the term "leading term" to refer to a monomial of largest excess in an element z of $VC(X)$; all other monomials comprising z desuspend farther than does the leading term.

Proposition 2.18. 1. ([23, 2.9]) *If a cycle of* $VC^2(S^{2n+1})$ *has order* p^f *in* $E_2^{2,t}(S^{2n+1})$ *and leading term* $h \otimes h^j\iota_{2n+1}$, *then* $j+\nu(|E_2^{2,t}(S^{2n+1})|) = f+n$.

2. ([20, 4.6]) *Let* $\nu=\nu(\sigma)$, *and let*

$$z = \epsilon v^{\sigma-e-1}h^e \otimes h + L \in VC^{2,2n+1+q\sigma}(S^{2n+1})$$

be a cycle with $\epsilon \in \mathbf{Z}_{(p)}$ *and* $\operatorname{exc}(L) < e-p+1 \le n-\nu$. *Then*

$$z = d(u\epsilon v^{\sigma-(e+\nu-p+2)}h^{e+\nu-p+2} + L'),$$

where u *is a unit in* $\mathbf{Z}_{(p)}$, *and* $\operatorname{exc}(L') < e+\nu-p+2$. *The same conclusion holds for* $z = \epsilon v^{\sigma-e+p-2}h \otimes h^{e-p+1} + L$.

We will need the following precise description of α_2.

Lemma 2.19. *The element* α_2 *which generates* $E_2^{1,2q+2n+1}(S^{2n+1})$ *is given by*

$$\alpha_2 = -d(v_1^2)/p = \tfrac{1}{p}(v^2-(v-ph)^2) = 2vh - ph^2 = hv + vh.$$

We will make repeated use of the following result, especially part (1).

Lemma 2.20. *Let $p = 3$. Then*

1. $\eta(v_1) = v_1 - 3h_1$
2. $\eta(v_2) = v_2 + 4v^3h - 18v^2h^2 + 35vh^3 - 24h^4 - 3h_2$
3. $\psi(h_1) = h_1 \otimes 1 + 1 \otimes h_1$
4. $\psi(h_2) = h_2 \otimes 1 + 1 \otimes h_2 + 4h^3 \otimes h + 6h^2 \otimes h^2 + 3h \otimes h^3 - vh \otimes h^2 - vh^2 \otimes h$

Proof. Parts (1) and (3) are standard, appearing in all referenced papers of the author and/or Bendersky. Part (2) is taken from Giambalvo's tables ([24]). Part (4) is derived from [13, 2.6i], using part (1) of this lemma several times to replace a v on the right (which is interpreted as $\eta(v)$) by $v - 3h$. Note, however, that the sum in [13, 2.6i] should be preceded by a minus sign. $\square$

The following result, proved in [23, 2.11,2.12,2.13], will be central to many of our calculations.

Lemma 2.21. 1. *If $n \geq 1$, then in $E_2^1(S^{2n+1})$, $h_1^p v_1 \equiv v_1^p h_1$ mod S^1.*
2. *$h_1^{n+p-1} \otimes h_1 \equiv -v_1^{p-1} h_1 \otimes h_1^n$ mod S^{2n-1} if $n > 1$;*
3. *$d(v_1^\ell h_1^{n+1}) \equiv -(\ell + n + 1) v_1^\ell h_1 \otimes h_1^n$ mod S^{2n-1}.*

3. New results about extensions

In this section, we show that $v_1^{-1}\pi_{2j-1}(X)$ is cyclic when X is a sphere bundle over a sphere with attaching map α_1 or α_2. This will be crucial to our proof of Theorem 1.1. We also determine the group structure of all groups $v_1^{-1}\pi_{2j-1}(X)$ when X is an exceptional Lie group for which the orders $|v_1^{-1}\pi_*(X)|$ have been determined.

The first result of this section is the following, in which $B_k(2n+1, 2n+kq+1)$ is an S^{2n+1}-bundle over $S^{2n+kq+1}$ with attaching map α_k.

Theorem 3.1. *Let $n > 1$, and $k = 1$ or 2. Then $v_{2j-1}(B_k(2n+1, 2n+kq+1))$ and $v_{2j}(B_k(2n+1, 2n+kq+1))$ are isomorphic cyclic p-groups with exponent*

$$\begin{cases} \min(n, 2 + \nu(j-n)) & \text{if } j \equiv n \bmod p(p-1) \\ \min(n + k(p-1), 2 + \nu(j - n - k(p-1))) & \text{if } j \equiv n \bmod (p-1) \\ & \quad \text{and } j \not\equiv n \bmod p(p-1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $B = B_k(2n+1, 2n+kq+1)$. The determination of $v_{2j}(B)$ when $k = 1$ was made in [13, 1.3(2)]. In [10, p.301], $v_{2j}(B)$ was determined when $k = 2$, $n = 4$, and $p = 3$. The argument there adapts to the general case in a straightforward fashion.

The cyclicity of $v_{2j-1}(B)$ when $k = 1$ is proved similarly to [23, p.613]. It is easy when $|v_{2j}(S^{2n+q+1})| = p^{n+p-1}$, for then $\partial : v_{2j}(S^{2n+q+1}) \to v_{2j-1}(S^{2n+1})$ is surjective, and so $v_{2j-1}(B) \approx v_{2j-1}(S^{2n+q+1})$ is cyclic.

Now consider the case when $|v_{2j}(S^{2n+q+1})| < p^{n+p-1}$. By Theorem 2.15(4), the class $d(p^{m-n-p}h^m)\iota_{2n+q+1}$ has order p in $E_2^2(S^{2n+q+1})$. Here m is an integer related to the stem of the class under consideration. Since ∂ annihilates this class, there is $w \in VC^2(S^{2n+1})$ such that $z := d(p^{m-n-p}h^m)\iota_{2n+q+1} - w$ is a cycle in $E_2^2(B)$. We wish to show that pz is the image of a generator of $E_2^2(S^{2n+1})$.

We use the formula

$$d(h^t\iota) = d(h^t)\iota + h^t d(\iota), \tag{3.2}$$

which was explained as [23, 5.4]. This implies that

$$pz = d(p^{m-n-p+1}h^m\iota_{2n+q+1}) - p^{m-n-p+1}h^m \otimes h\iota_{2n+1} - pw.$$

The first term is a boundary. Using Lemma 2.20, the second term is, mod terms that desuspend below S^{2n+1}, $-v_1^{m-n-p+1}h^{n+p-1} \otimes h\iota_{2n+1}$, and by Lemma 2.21(2), this is, mod lower terms, $v_1^{m-n-p+1}h \otimes h^n\iota_{2n+1}$, which is the leading term of a generator of $E_2^2(S^{2n+1})$, by Theorem 2.15 or Proposition 2.18. Also, as we shall show in the next paragraph, pw desuspends to S^{2n-1}. Since the double suspension from $E_2^2(S^{2n-1})$ to $E_2^2(S^{2n+1})$ is not surjective, this implies that pz is the image of a generator of $E_2^2(S^{2n+1})$.

One way to see that w can be chosen so that pw desuspends to S^{2n-1} is to note that $pz = d(p^{m-n-p+1}h^m)\iota_{2n+q-1} - pw$ is a cycle in $E_2^2(B(2n-1, 2n+q-1))$; i.e., multiplying by p allows you to double desuspend the whole equation.

The argument when $k = 2$ is very similar. We will have $w \in VC^2(S^{2n+1})$ satisfying that $z := d(p^{m-n-2p+1}h^m)\iota_{2n+2q+1} - w$ is a cycle in $E_2^2(B)$, and, as in the previous paragraph, w can be chosen so that pw double desuspends. We obtain

$$pz = d(p^{m-n-2p+2}h^m\iota_{2n+2q+1}) - p^{m-n-2p+2}h^m \otimes \alpha_2\iota_{2n+1} - pw.$$

The first term is a boundary, the last term desuspends, while the middle term is, mod terms that desuspend, $v_1^{m-n-2p+2}h^{n+2p-2} \otimes (2vh - ph^2)\iota_{2n+1}$. The term with ph^2 desuspends, while the first term is, by 2.21(2), $2v_1^{m-n-p+1}h^{n+p-1} \otimes h\iota_{2n+1}$, which is the leading term of a generator of $E_2^2(S^{2n+1})$. The class $\{pz\}$ generates $E_2^2(S^{2n+1})$ by Proposition 2.18(1). □

In [13] and [10], sphere bundles X of the type covered by Theorem 3.1 occurred as factors in product decompositions of exceptional Lie groups (localized at a prime p). In those papers, we merely asserted the order of the groups $v_1^{-1}\pi_{2j-1}(X)$, but we can now declare that they are cyclic. There are a few other cases of factors Y of exceptional Lie groups for which only the order but not the group structure of $v_1^{-1}\pi_{2j-1}(Y)$ was given in [13], but we can now complete the determination of the v_1-periodic homotopy of all torsion-free exceptional Lie groups by giving the group structure in these cases. The following result handles all of these, and those left unresolved in [10].

Proposition 3.3. 1. *The 3-primary groups $v_{2j-1}(B(11,15))$ and $v_{2j-1}(E_6/F_4)$, which occur in* [10, 1.2], *are cyclic.*

2. *The factors $B(2n+1, 2n+q+1)$ which occur in G_2 for $p = 5$, F_4 and E_6 for $5 \le p \le 11$, E_7 for $11 \le p \le 17$ and for $p = 5$, and E_8 for $11 \le p \le 29$, as listed in* [13, 1.1], *have $v_{2j-1}(B)$ cyclic of order given in* [13, 1.3(2)].
3. *The spaces $B(11,23,35)$ and $B(23,35,47,59)$, which occur as factors in 7-primary E_7 and E_8, respectively, have $v_{2j-1}(B)$ cyclic of order given in* [13, 1.4].
4. *The spaces $B(3,11,19,27,35)$, $B(3,15,27)$, and $B(3,15,27,39)$, which occur as factors of 5-primary E_7, 7-primary E_7, and 7-primary E_8, respectively, have $v_{2j-1}(B) \approx \mathbf{Z}/p \oplus \mathbf{Z}/p^{e-1}$, where e is the number given in* [13, 1.4].

Proof. The first two parts are immediate from Theorem 3.1. The first space in part 3 is a factor of $SU(18)$, and in the notation of [23, 1.5] it has $N = 5$ and $i = 2$. By [23, 1.9], its $v_{2j-1}(-)$-groups are cyclic. Similarly, the second space in part 3 is a quotient of a factor $B = B(11,23,35,47,59)$ of $SU(30)$. This factor B has $N = 5$ and $i = 4$ in the notation of [23, 1.5], and hence its groups $v_{2j-1}(-)$ are cyclic by [23, 1.9]. Thus so are the groups of the desired space $B(23,35,47,59)$, since $v_{2j-2}(S^{11}) = 0$ for values of j under consideration.

The spaces in part 4 are also factors of $SU(n)$ and hence are covered by [23, 1.9]. In the notation of [23, 1.5], these three spaces each have $N = 1$, while $i = 4$, 2, and 3, respectively, and $\hat{m} > 0$. Thus their $v_{2j-1}(-)$ has a split $\mathbf{Z}/p$ by [23, 1.9]. □

4. Discussion of E_7/F_4

In this section we sketch a natural approach to Theorem 1.1. Although we will not follow it exactly, it is helpful in understanding the approach which we do employ. Also, the result here about the convergence of the v_1-periodic UNSS for E_7/F_4 will play a key role in our later deduction of $v_1^{-1}\pi_*(E_7)$. Throughout the remainder of the paper, we will have $p = 3$.

The fibration

$$F_4 \to E_7 \to E_7/F_4 \tag{4.1}$$

induces a long exact sequence of v_1-periodic homotopy groups. The groups $v_*(F_4)$ were computed in [10], while $v_*(E_7/F_4)$ could be computed by the methods of this paper. Then we would need to determine the boundary homomorphism and extensions in the exact sequence associated to (4.1). This determination is complicated by the fact that the Bockstein β is nonzero in $H^*(F_4;\mathbf{Z}/3)$, which causes $BP_*(F_4)$ to be not a free BP_*-module, and therefore the UNSS of F_4 cannot be calculated directly by known methods. (In [10], $v_*(F_4)$ was determined by a combination of topological and UNSS methods.) Moreover, applying Ω to the fibration does not help much, because $BP_*(\Omega E_7)$ is not a free commutative algebra, and so we cannot apply Theorem 2.6 to compute its v_1-periodic UNSS. Hence UNSS methods cannot be used directly to analyze the exact sequence in $v_*(-)$ associated to (4.1).

Our proof could be expedited slightly if we were assured of the validity of the following conjecture, due to Mimura.

Conjecture 4.2. *Localized at* 3*,* E_7/F_4 *is spherically resolved by spheres of dimension* 19*,* 27*, and* 35*, and attaching maps* α_2*. That is, there is a fibration* $S^{19} \to E_7/F_4 \to B_2(27,35)$ *and a fibration* $S^{27} \to B_2(27,35) \to S^{35}$*, with attaching maps from* 19 *to* 27 *and from* 27 *to* 35 *both equal to the element* α_2 *which generates* $\pi_7(S^0)_{(3)} \approx \mathbf{Z}/3$*.*

Although we cannot use this proposed topological description of E_7/F_4, we can say enough about this space to compute its v_1-periodic UNSS and prove that it converges to $v_*(E_7/F_4)$. However, the specific results of this computation will not be needed for the reasons cited earlier in this section, and the methods will be applied again in computing the v_1-periodic UNSS of the space Y_7, which will be our approach to $v_*(E_7)$, and so we shall wait until the next section to use them.

The following first steps toward proving Conjecture 4.2 will be useful to us later. They were pointed out by Mimura.

Proposition 4.3. (a) *$H^*(E_7/F_4;\mathbf{Z})$ is an exterior algebra on classes of dimension 19, 27, and 35.* (b) *The 35-skeleton of E_7/F_4 is $S^{19}\cup_{\pm\alpha_2} e^{27}\cup_{\alpha_2} e^{35}$, where α_2 generates $\pi_7(S^0)_{(3)}\approx \mathbf{Z}/3$.*

The proof of this proposition requires the following result of Kono and Mimura.

Proposition 4.4. ([29]) *There is an algebra isomorphism*

$$H^*(E_7;\mathbf{Z}_3)\approx \mathbf{Z}_3[e_8]/(e_8^3)\otimes\Lambda[e_3,e_7,e_{11},e_{15},e_{19},e_{27},e_{35}]$$

with only nonzero action of β or $\mathcal{P}^{p^r}$ on generators given by $\beta e_7=e_8$, $\beta e_{15}=-e_8^2$, $\mathcal{P}^1e_3=e_7$, $\mathcal{P}^1e_{11}=e_{15}$, $\mathcal{P}^1e_{15}=\pm e_{19}$, $\mathcal{P}^3e_7=e_{19}$, $\mathcal{P}^3e_{15}=e_{27}$.

Proof of Proposition 4.3. Part (a) was proved in [30, 9.4]. To prove part (b), let Φ denote the secondary cohomology operation associated with the relation $\mathcal{P}^1\beta\mathcal{P}^1-\beta\mathcal{P}^2-\mathcal{P}^2\beta=0$. This secondary operation detects the map α_2 and satisfies $\mathcal{P}^3=\mathcal{P}^1\Phi$. (See [29, p.353].) In [29, 7.2], it is shown that $\Phi(\tilde e_{27})=\tilde e_{35}$ in $H^*(\tilde E_7)$, where $\tilde E_7$ denotes the fiber of $E_7\to K(\mathbf{Z},3)$, from which it follows that $\Phi(e_{27})=e_{35}$ in $H^*(E_7)$.

Since $\mathcal{P}^1e_{15}=\pm e_{19}$ and $\mathcal{P}^3e_{15}=e_{27}$ in $H^*(E_7)$, we can use a dual relation $\mathcal{P}^3=\Phi\mathcal{P}^1$ to deduce that $\Phi(e_{19})=\pm e_{27}$. The dual relation is deduced by applying the original relation in the S-dual, and then noting that $\mathcal{P}^1$, $\mathcal{P}^3$, and Φ are all self-dual. Here duality is given by the antiautomorphism of the Steenrod algebra, while Φ is self-dual since it is defined by a symmetric Adem relation involving self-dual terms. □

We close this section by proving the following result, which will be crucial for us, since we will use it later to deduce that the v_1-periodic UNSS of Y_7 converges to $v_*(Y_7)$.

Theorem 4.5. *The v_1-periodic UNSS of E_7/F_4 converges to $v_*(E_7/F_4)$. Indeed, Theorem 2.14 holds if $X=E_7/F_4$.*

Note that this result would be immediate from 2.14 if we knew that Conjecture 4.2 were true. Instead, we must call upon the following result, which was proved by Bendersky and Thompson at the request of the author. The statement and proof of this result rely heavily on the work of Bousfield ([17]), who defined a space to be K/p_*-durable when its K/p_*-localization map induces an isomorphism in $v_*(-)$.

Theorem 4.6. ([15]) *Suppose X is a K/p_*-durable space with $K^*(X;\widehat{\mathbf{Z}}_p)$ isomorphic as a $\mathbf{Z}/2$-graded p-adic λ-ring to $\widehat\Lambda(M)$, where $M=M_n$ is a p-adic Adams module which admits a sequence of epimorphisms of p-adic Adams modules*

$$M_n\xrightarrow{p_n}M_{n-1}\xrightarrow{p_{n-1}}\cdots\xrightarrow{p_2}M_1=M(2m_1+1)$$

with $\ker(p_i)=M(2m_i+1)$ for $2\le i\le n$. Here $M(2m+1)\approx K^*(S^{2m+1};\widehat{\mathbf{Z}}_p)$ as a p-adic Adams module, and $\widehat\Lambda(M)$ denotes the exterior algebra on M. Then the (BP-based) v_1-periodic UNSS of X converges to $v_*(X)$.

Actually, what is proved in [15] is that if $X\hat{}$ denotes the K/p-completion, then $X\to X\hat{}$ induces an isomorphism in $v_*(-)$. In [14], it is proved that the v_1-periodic UNSS converges to $v_*(X\hat{})$, which then implies Theorem 4.6. Theorem 4.5 is an immediate consequence of Theorem 4.6 and the following two results.

Theorem 4.7. *There is an isomorphism of* $\mathbf{Z}/2$*-graded* p*-adic* λ*-rings*

$$K^*(E_7/F_4;\widehat{\mathbf{Z}}_p) \approx \widehat{\Lambda}(M_3),$$

with short exact sequences of p*-adic Adams modules*

$$0 \to M(35) \to M_3 \to M_2 \to 0 \quad \textit{and} \quad 0 \to M(27) \to M_2 \to M(19) \to 0.$$

Theorem 4.8. E_7/F_4 *is* $K/3_*$*-durable.*

Proof of Theorem 4.7. We use Proposition 4.3 to give the E_2-term of the Atiyah-Hirzebruch spectral sequence converging to $K^*(E_7/F_4;\widehat{\mathbf{Z}}_p)$ as $\Lambda[x_{19},x_{27},x_{35}] \otimes K\widehat{\mathbf{Z}}_p^*$. The spectral sequence collapses to yield the claimed exterior algebra as $K^*(E_7/F_4;\widehat{\mathbf{Z}}_p)$. This collapsing can be deduced from Yagita's result ([34]) that there is a 3-local isomorphism

$$BP^*(E_7) \approx BP^*(F_4) \otimes \Lambda[19,27,35],$$

or from Snaith's result ([32]) that the spectral sequence

$$\mathrm{Tor}_{R(G)}(\mathbf{Z}, R(H)) \Rightarrow K^*(G/H)$$

collapses.

The claim about the decomposition of M_3 as a p-adic Adams module will follow once we show that the generators of the exterior algebra $K^1(E_7/F_4)$ satisfy $\psi^k(x_{35}) = k^{17}x_{35}$, $\psi^k(x_{27}) = k^{13}x_{27} + \alpha_1 x_{35}$, and $\psi^k(x_{19}) = k^9 x_{19} + \alpha_2 x_{27} + \alpha_3 x_{35}$ for some integers α_1, α_2, and α_3. Note that $K^1(E_7/F_4)$ is spanned by x_{19}, x_{27}, x_{35}, and $x_{19}x_{27}x_{35}$. We will show that the top cell of E_7/F_4, which corresponds to this product class, splits off stably, and so cannot be involved in Adams operations on the lower classes. Then the formula for the Adams operations follows from the inclusions $S^{19} \to E_7/F_4$, $S^{27} \to (E_7/F_4)/S^{19}$, and $S^{35} \to (E_7/F_4)/(E_7/F_4)^{(27)}$.

To prove the stable splitting, we argue similarly to [22, 1.1]. By [2, 3.3], the S-dual of the manifold E_7/F_4 is the Thom spectrum of its stable normal bundle. However, $\widetilde{KO}(E_7/F_4)_{(3)} = 0$, since E_7/F_4 has no cells whose dimension is a multiple of 4. Thus the bottom class splits off the Thom spectrum of the stable normal bundle, and dually the top cell stably splits off the manifold itself. □

The following proof is due to Pete Bousfield.

Proof of Theorem 4.8. In [17], Bousfield utilizes a functor Φ from spaces to spectra, which he had introduced in earlier papers. A map f induces an isomorphism in $v_*(-)$ if and only if $\Phi(f)$ is an equivalence. Let $X = E_7/F_4$, and consider the commutative diagram

$$\begin{array}{ccccc}
\Phi(F_4) & \longrightarrow & \Phi(E_7) & \longrightarrow & \Phi(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Phi((F_4)_{K/p}) & \longrightarrow & \Phi((E_7)_{K/p}) & \longrightarrow & \Phi(X_{K/p})
\end{array}$$

Since Φ preserves fibrations, the top row is a fibration, and since [17, 7.8] states that H-spaces are K/p_*-durable, the first two vertical arrows are equivalences. We will be done by the 5-lemma once we show that the bottom row is a fibration.

By [17, 6.3], $K^*(G;\widehat{\mathbf{Z}}_p) \approx \widehat{\Lambda}(P_G)$, where $P_G = PK^1(G;\widehat{\mathbf{Z}}_p))$, for $G = F_4$ or E_7, and by [17, 8.1] $\Phi(G_{K/p}) \simeq \Phi G$ is a $K\widehat{\mathbf{Z}}_p^*$-Moore spectrum $\mathcal{M}(P_G/\psi^p, 1)$, where

P_G/ψ^p is the quotient by the injective action of the Adams operation. Similarly, by Theorem 4.7,

$$K^*(X_{K/p};\widehat{\mathbf{Z}}_p) \approx K^*(X;\widehat{\mathbf{Z}}_p) \approx \widehat{\Lambda}(M_3),$$

and, since $X_{K/p}$, being K/p-local, is certainly K/p_*-durable, we can apply [17, 8.1] to obtain $\Phi(X_{K/p}) \simeq \mathcal{M}(M_3/\psi^p)$. There is a short exact sequence of Adams modules, (e.g. from [34])

$$0 \to M_3 \to PK^1(E_7) \to PK^1(F_4) \to 0$$

and hence a fiber sequence

$$\mathcal{M}(P_{F_4}/\psi^p, 1) \to \mathcal{M}(P_{E_7}/\psi^p, 1) \to \mathcal{M}(M_3/\psi^p, 1)$$

which is the bottom row of the commutative diagram considered above, showing that it is a fibration, as desired. □

5. E_2 of periodic UNSS of $\Omega E_7/Sp(2)$

In this long section, we calculate the periodic UNSS of $Y_7 := \Omega E_7/Sp(2)$. In Section 7, we perform the transition from these results to $v_1^{-1}\pi_*(E_7)$.

We begin by recalling the following result of Harper, which we used in [10].

Proposition 5.1. ([27, 4.4.1]) *There is a* 3*-equivalence*

$$F_4 \approx K \times B(11,15),$$

where K is a finite mod 3 *H-space satisfying*

$$H^*(K;F_3) = \Lambda(x_3,x_7)\otimes F_3[x_8]/(x_8^3),$$

with $x_7 = \mathcal{P}^1 x_3$ and $x_8 = \beta x_7$. Also, $B(11,15)$ is an S^{11}-bundle over S^{15} with $\mathcal{P}^1 x_{11} = x_{15}$. Moreover, there is a fibration $B(3,7) \to K \to W$, where W is the Cayley plane, and a fibration $S^7 \to \Omega W \to \Omega S^{23}$.

Because of the torsion in $H^*(K;\mathbf{Z})$, and hence in $H^*(E_7;\mathbf{Z})$, we will work with loop spaces, and use the following result of Hamanaka and Hara ([26]).

Proposition 5.2. *The mod* 3 *homology as Hopf algebras over the Steenrod algebra satisfies*

$$\begin{aligned} H_*(\Omega F_4) &\approx F_3[t_2,t_6,t_{10},t_{14},t_{22}]/(t_2^3)\\ H_*(\Omega E_7) &\approx F_3[t_2,t_6,t_{10},t_{14},t_{18},t_{22},t_{26},t_{34}]/(t_2^3),\end{aligned}$$

with the only nonzero reduced coproducts being

$$\overline{\phi}(t_6) = -t_2^2\otimes t_2 - t_2\otimes t_2^2$$

and

$$\begin{aligned}\overline{\phi}(t_{18}) &= t_2^2t_6^2\otimes t_2 + t_2t_6^2\otimes t_2^2 - t_6^2\otimes t_6 - t_2^2t_6\otimes t_2t_6\\ &\quad -t_2t_6\otimes t_2^2t_6 - t_6\otimes t_6^2 + t_2^2\otimes t_2t_6^2 + t_2\otimes t_2^2t_6^2.\end{aligned}$$

The only nonzero action of dual Steenrod operations $\mathcal{P}_*^{3^r}$ are $\mathcal{P}_*^1(t_6) = t_2$, $\mathcal{P}_*^1(t_{14}) = t_{10}$, $\mathcal{P}_*^1(t_{18}) = \epsilon t_{14} - t_2t_6^2$, $\mathcal{P}_*^1(t_{22}) = \kappa t_6^3$, $\mathcal{P}_*^1(t_{26}) = \epsilon t_{22}$, $\mathcal{P}_*^1(t_{34}) = -\epsilon t_{10}^3$, $\mathcal{P}_*^3(t_{18}) = t_6$, $\mathcal{P}_*^3(t_{26}) = t_{14}$, and $\mathcal{P}_*^3(t_{34}) = t_{22}$. Here $\epsilon = \pm 1$ and $\kappa = \pm 1$.

Because of the relation $t_2^3 = 0$ in $H_*(\Omega E_7)$, Theorem 2.6 does not apply to $X = \Omega E_7$. Instead, we will work with the space Y_7 defined in the following theorem. We begin by noting (see [10, p.296]) that the space $B(3,7)$ which occurs in 5.1 is 3-equivalent to $Sp(2)$.

Theorem 5.3. *Let $E_7/Sp(2)$ denote the quotient of the group inclusion $Sp(2) \to F_4 \to E_7$, and let $Y_7 = \Omega E_7/Sp(2)$. Then*

$$H_*(Y_7; G) \approx \Lambda[x_7] \otimes G[x_{10}, x_{14}, x_{18}, x_{22}, x_{26}, x_{34}]$$

for $G = \mathbf{Z}/3$ or $\mathbf{Z}_{(3)}$.

Proof. There is a commutative diagram of fibrations

$$\begin{array}{ccccc} Y_7 & \to & B(3,7) & \to & E_7 \\ \downarrow & & \downarrow & & \downarrow{=} \\ \Omega E_7/F_4 & \to & F_4 & \to & E_7, \end{array} \tag{5.4}$$

and this, together with the fibration $B(3,7) \to F_4 \to W \times B(11,15)$, which is a consequence of 5.1, implies there is a fibration

$$\Omega W \times \Omega B(11,15) \to Y_7 \to \Omega E_7/F_4. \tag{5.5}$$

The last fibration in 5.1 determines $H_*(\Omega W)$, and the Serre spectral sequence of (5.5) collapses, yielding the claim of the theorem. The collapsing is proved by observing that the only possible differential on one of the three polynomial generators is $d_{17}(x_{18}) = \epsilon x_7 \otimes x_{10}$, but this has $\epsilon = 0$ by consideration of the map from (5.5) to the fibration

$$\Omega W \times \Omega B(11,15) \to B(3,7) \to F_4.$$

□

We easily obtain the following consequence.

Corollary 5.6. *$BP_*(Y_7)$ is a free commutative algebra on classes x_7, x_{10}, x_{14}, x_{18}, x_{22}, x_{26}, and x_{34}, with $x_i \in BP_i(Y_7)$.*

Proof. By [1, 12.1], the rationalization of Y_7 is homotopy equivalent to $K(Q,7) \times K(Q,10) \times \cdots \times K(Q,34)$. Any differentials in the Atiyah-Hirzebruch spectral sequence

$$\Lambda[x_7] \otimes \mathbf{Z}_{(3)}[x_{10}, x_{14}, x_{18}, x_{22}, x_{26}, x_{34}] \otimes BP_* \Rightarrow BP_*(Y_7)$$

must be seen rationally, and hence must be zero. That $x_7^2 = 0$ is deduced from the inclusion $S^7 \to Y_7$. □

By Theorem 2.6, the UNSS of Y_7 can be calculated as the homology of the unstable cobar complex. This complex splits as the direct sum of the unstable cobar complex for S^7 plus the even-dimensional complex. That is, we have

$$E_2^{s,t}(Y_7) \approx \begin{cases} E_2^{s,t}(S^7) & \text{if } t \text{ is odd} \\ \mathrm{Ext}_{\mathcal{V}}^{s,t}(BP_*\langle x_{10}, x_{14}, x_{18}, x_{22}, x_{26}, x_{34}\rangle) & \text{if } t \text{ is even} \end{cases} \tag{5.7}$$

Our work in this section will go into computing

$$v_1^{-1}\,\mathrm{Ext}_{\mathcal{V}}^{s,t}(BP_*\langle x_{10}, x_{14}, x_{18}, x_{22}, x_{26}, x_{34}\rangle).$$

This is the v_1-periodic Ext which forms the E_2-term of the v_1-periodic UNSS of Y_7. In Section 7, we will use Theorem 4.6 to show that this spectral sequence converges to $v_*(Y_7)$. Throughout the remainder of the paper, E_2 **and** $\mathrm{Ext}_{\mathcal{V}}$ **will always refer to their** v_1**-periodic versions**, unless explicitly stated to the contrary.

To compute the homology of the unstable cobar complex of Y_7, we will utilize exact sequences in $\mathrm{Ext}_{\mathcal{V}}(-)$ induced by the injective extension sequences

(5.8) $$A(26) \to A(26,34) \to A(34),$$

(5.9) $$A(18) \to A(18,26,34) \to A(26,34),$$

and

(5.10) $$A(10,14) \otimes A(22) \to BP_{\mathrm{ev}}(Y_7) \to A(18,26,34).$$

Each of these $A(-)$ is the subquotient of $BP_*(Y_7)$ on the generators of the indicated dimensions. Each has an induced Γ-coaction. The sequence (5.10) is closely related to the fibration

$$F_4 \to E_7 \to E_7/F_4,$$

with $F_4 \approx B(11,15) \times K$.

By [9, 4.3], each of these three injective extension sequences yields a long exact sequence when ordinary (unlocalized) $\mathrm{Ext}_{\mathcal{V}}(Q(-))$ is applied, and these Ext-groups are the homology of the associated unstable cobar complexes. The v_1-periodic E_2-term is the direct limit of a direct system of v_1-power morphisms, and these commute with the morphisms in the exact sequences just described. Since the direct limit of exact sequences is exact, we obtain that there is an exact sequence of v_1-periodic E_2-terms. As observed after Theorem 2.14, we can still work with the unstable cobar complex, as long as we restrict attention to v_1-periodic classes. We will abbreviate $\mathrm{Ext}_{\mathcal{V}}(Q(A(n_1,\cdots,n_k)))$ as $E_2(n_1,\cdots,n_k)$, and the associated unstable cobar complex as $C(n_1,\cdots,n_k)$.

In order to analyze ∂ in the long exact Ext sequences, we will need the following crucial result about the Γ-coaction.

Proposition 5.11. *If M is a Γ-comodule which as a BP_*-module is free on x_{10}, x_{14}, x_{18}, x_{26}, and x_{34}, and if*

$$\begin{aligned}
\psi(x_{34}) &= 1\otimes x_{34} + \alpha_2 \otimes x_{26} + T_1 \otimes x_{18} + T_2 \otimes x_{14} + T_3 \otimes x_{10}\\
\psi(x_{26}) &= 1\otimes x_{26} + \alpha_2 \otimes x_{18} + T_4 \otimes x_{14} + T_5 \otimes x_{10}\\
\psi(x_{18}) &= 1\otimes x_{18} + \alpha_1 \otimes x_{14} + T_6 \otimes x_{10}\\
\psi(x_{14}) &= 1\otimes x_{14} + \alpha_1 \otimes x_{10},
\end{aligned}$$

then $T_6 = \frac{1}{2}h^2$, and, mod terms that desuspend lower than does the indicated term, $T_1 \equiv \frac{1}{2}v^2h^2$, $T_2 \equiv -5vh^4$, $T_3 \equiv \frac{1}{4}vh^5$, $T_4 \equiv h^3$, and $T_5 \equiv \frac{1}{4}vh^3$.

This proposition will be applied when M is a quotient of $BP_*(Y_7)$. The α_2-terms in $\psi(x_{26})$ and $\psi(x_{34})$ are present there by Proposition 4.3, since α_2 is the cycle which detects the homotopy class α_2. We will see after the proof that our application of this proposition to computing the homology of the unstable cobar complex would not be affected if a unit coefficient were present on α_2. Similarly, the α_1-terms in $\psi(x_{18})$ and $\psi(x_{14})$ are present because of $\mathcal{P}^1_*(t_{18})$ and $\mathcal{P}^1_*(t_{14})$ in 5.2, and the homology application would not be affected if they were multiplied by a unit.

Proof. We begin with the determination of T_1. Using also that $\psi x_{26} = 1\otimes x_{26} + \alpha_2\otimes x_{18}$ mod lower terms, the coassociativity formula $(\psi\otimes 1)\psi(x_{34}) = (1\otimes\psi)\psi(x_{34})$ implies that $\psi(\alpha_2) = \alpha_2\otimes 1 + 1\otimes\alpha_2$ and $\psi(T_1) = T_1\otimes 1 + 1\otimes T_1 + \alpha_2\otimes\alpha_2$. Now, α_2 is given in Lemma 2.19, and one can verify that it is primitive. Let $\overline{\psi}$ be the reduced coproduct in BP_*BP, defined by $\psi(y) = y\otimes 1 + 1\otimes y + \overline{\psi}(y)$. We use the condition that $\overline{\psi}(T_1) = \alpha_2\otimes\alpha_2$ to find T_1.

First, using Lemma 2.19 and Lemma 2.20(1), we compute

$$\begin{aligned}\alpha_2\otimes\alpha_2 &= (2vh-3h^2)\otimes(2vh-3h^2)\\ &= 4vh\otimes vh - 6h^2\otimes vh - 6vh\otimes h^2 + 9h^2\otimes h^2\\ &= 4v^2h\otimes h - 18vh^2\otimes h + 18h^3\otimes h - 6vh\otimes h^2 + 9h^2\otimes h^2.\end{aligned}$$

Now T_1 must be a combination of the following five terms, whose $\overline{\psi}$ are listed.

$$\begin{aligned} h^4 &\mapsto 4h^3\otimes h + 6h^2\otimes h^2 + 4h\otimes h^3.\\ vh^3 &\mapsto 3vh^2\otimes h + 3vh\otimes h^2 + v\otimes h^3 - 1\otimes vh^3\\ &= 3vh^2\otimes h + 3vh\otimes h^2 + 3h\otimes h^3.\\ v^2h^2 &\mapsto 2v^2h\otimes h + v^2\otimes h^2 - 1\otimes v^2h^2\\ &= 2v^2h\otimes h + 6vh\otimes h^2 - 9h^2\otimes h^2.\\ v^3h &\mapsto v^3\otimes h - 1\otimes v^3h = 9v^2h\otimes h - 27vh^2\otimes h + 27h^3\otimes h.\\ h_2 &\mapsto 4h^3\otimes h + 6h^2\otimes h^2 + 3h\otimes h^3 - vh\otimes h^2 - vh^2\otimes h.\end{aligned}\tag{5.12}$$

We solve a system of linear equations for the coefficients of these five terms, to see what combination T_1 can have $\overline{\psi}(T_1) = \alpha_2\otimes\alpha_2$, as required. We find that the desired term T_1 is given by

$$\begin{aligned}T_1 = \tfrac{9}{2}h^4 - 6vh^3 + 2v^2h^2 + c_1(-3h^4 + vh^3 + 3h_2)\\ + c_2(-\tfrac{27}{4}h^4 + 9vh^3 - \tfrac{9}{2}v^2h^2 + v^3h),\end{aligned}$$

with c_1 and c_2 in $\mathbf{Z}_{(3)}$. Replacing $3h$ by $v-\eta(v)$ at several places, this simplifies to

$$T_1 = \tfrac{1}{2}v^2h^2 + L,$$

where L desuspends to S^3.

The other T_i's are determined similarly. Coassociativity implies

$$\begin{aligned}\overline{\psi}(T_6) &= \alpha_1\otimes\alpha_1\\ \overline{\psi}(T_4) &= \alpha_2\otimes\alpha_1\\ \overline{\psi}(T_5) &= \alpha_2\otimes T_6 + T_4\otimes\alpha_1\\ \overline{\psi}(T_2) &= \alpha_2\otimes T_4 + T_1\otimes\alpha_1\\ \overline{\psi}(T_3) &= \alpha_2\otimes T_5 + T_1\otimes T_6 + T_2\otimes\alpha_1.\end{aligned}\tag{5.13}$$

That T_6 must equal $\frac{1}{2}h^2$ is easily determined (since $\alpha_1 = -h$). To determine T_4, we write $\alpha_2\otimes\alpha_1 = -2vh\otimes h + 3h^2\otimes h$, and note that $\overline{\psi}$ acts as follows:

$$\begin{aligned}h^3 &\mapsto 3h^2\otimes h + 3h\otimes h^2\\ vh^2 &\mapsto 2vh\otimes h + 3h\otimes h^2\\ v^2h &\mapsto 6vh\otimes h - 9h^2\otimes h\end{aligned}$$

Solving a system of equations for the coefficients of h^3, vh^2, and v^2h yields

$$T_4 = h^3 - vh^2 + c(3h^3 - 3vh^2 + v^2h), \tag{5.14}$$

with $c \in \mathbf{Z}_{(3)}$. All terms except the first are defined on S^5, and so T_4 is as claimed.

We must have $\overline{\psi}(T_5) = vh \otimes h^2 - \frac{3}{2}h^2 \otimes h^2 - h^3 \otimes h + vh^2 \otimes h$. The terms of which T_5 is a linear combination are the same as those in T_4, which were listed with their $\overline{\psi}(-)$ in (5.12). Solving this system of equations yields that the combination whose $\overline{\psi}(-)$ is that required of T_5 can be

$$-\tfrac{1}{4}h^4 + \tfrac{1}{3}vh^3 + c_1(-h^4 + \tfrac{1}{3}vh^3 + h_2) + c_2(-\tfrac{27}{4}h^4 + 9vh^3 - \tfrac{9}{2}v^2h^2 + v^3h),$$

for any c_1 and c_2 in $\mathbf{Z}_{(3)}$. However, fractions with 3 in the denominator do not lie in $\mathbf{Z}_{(3)}$. The only way to prevent this is to specify that c_1 must be of the form $-1 + 3k$, with $k \in \mathbf{Z}_{(3)}$. This yields

$$T_5 = \tfrac{3}{4}h^4 - h_2 + k(-3h^4 + vh^3 + 3h_2) + L,$$

where L (the c_2-term) desuspends to S^3. In two places, we replace $3h^4$ by $vh^3 - h^3v$, yielding

$$T_5 = \tfrac{1}{4}vh^3 + L', \tag{5.15}$$

as desired. In our determination of T_5, we should also take into account the homogeneous part of (5.14), as it contributes to the $T_4 \otimes \alpha_1$-term of $\overline{\psi}(T_5)$. When the resulting equations are solved, we obtain an additional homogeneous part of T_5, equal to

$$c'(\tfrac{3}{4}h^4 - vh^3 + \tfrac{1}{2}v^2h^2) = c'(-\tfrac{3}{4}vh^3 - \tfrac{3}{4}h^3v + \tfrac{1}{2}v^2h^2),$$

which desuspends farther than the leading term of (5.15). Thus T_5 is as claimed.

Similarly, by (5.13) we must have (mod homogeneous terms that will be considered below)

$$\begin{aligned}
\overline{\psi}(T_2) &= (2vh - 3h^2) \otimes (h^3 - vh^2) - (\tfrac{9}{2}h^4 - 6vh^3 + 2v^2h^2) \otimes h \\
&= 2vh \otimes h^3 - 3h^2 \otimes h^3 - 2v^2h \otimes h^2 + 9vh^2 \otimes h^2 - 9h^3 \otimes h^2 \\
&\quad -\tfrac{9}{2}h^4 \otimes h + 6vh^3 \otimes h - 2v^2h^2 \otimes h.
\end{aligned}$$

The terms that can comprise T_2 are listed below, with their $\overline{\psi}$.

$$\begin{array}{rcl}
h^5 & \mapsto & 5h \otimes h^4 + 10h^2 \otimes h^3 + 10h^3 \otimes h^2 + 5h^4 \otimes h \\
vh^4 & \mapsto & 3h \otimes h^4 + 4vh \otimes h^3 + 6vh^2 \otimes h^2 + 4vh^3 \otimes h \\
v^2h^3 & \mapsto & -9h^2 \otimes h^3 + 6vh \otimes h^3 + 3v^2h \otimes h^2 + 3v^2h^2 \otimes h \\
v^3h^2 & \mapsto & 27h^3 \otimes h^2 - 27vh^2 \otimes h^2 + 9v^2h \otimes h^2 + 2v^3h \otimes h \\
v^4h & \mapsto & -81h^4 \otimes h + 108vh^3 \otimes h - 54v^2h^2 \otimes h + 12v^3h \otimes h \\
vh_2 & \mapsto & 3vh \otimes h^3 + 6vh^2 \otimes h^2 + 4vh^3 \otimes h - v^2h \otimes h^2 - v^2h^2 \otimes h + 3h_1 \otimes h_2 \\
h_1h_2 & \mapsto & 3h \otimes h^4 + 9h^2 \otimes h^3 + 10h^3 \otimes h^2 + 4h^4 \otimes h - vh \otimes h^3 \\
 & & -2vh^2 \otimes h^2 - vh^3 \otimes h + h_1 \otimes h_2 + h_2 \otimes h_1 \\
v_2h_1 & \mapsto & 24h^4 \otimes h - 35vh^3 \otimes h + 18v^2h^2 \otimes h - 4v^3h \otimes h + 3h_2 \otimes h_1
\end{array}$$

We solve a system of equations to find the combination of these terms having $\overline{\psi}$ as desired. We obtain

$$\begin{aligned} T_2 &= -\tfrac{9}{10}h^5+\tfrac{3}{2}vh^4-\tfrac{2}{3}v^2h^3+c_1(\tfrac{81}{5}h^5-27vh^4+18v^2h^3-6v^3h^2+v^4h) \\ &\quad +c_2(-\tfrac{12}{5}h^5+7vh^4-\tfrac{17}{3}v^2h^3+2v^3h^2+vh_2-3h_1h_2+v_2h_1). \end{aligned}$$

As in the previous case, in order to prevent 3 in a denominator, we choose $c_2 = -1+3c$. This yields

$$T_2 = \tfrac{3}{2}h^5 - \tfrac{11}{2}vh^4 + 5v^2h^3 - 2v^3h^2 - vh_2 + 3h_1h_2 - v_2h_1$$

plus two homogeneous terms which are defined on S^7. The first two terms in T_2 combine to $\frac{1}{2}vh^4 - \frac{1}{2}h^4v - \frac{11}{2}vh^4$, and so, mod terms that are defined on S^7, we have $T_2 \equiv -5vh^4$, as claimed. We have omitted here consideration of homogeneous parts of T_4 and T_1 already obtained. These yield additional homogeneous terms in T_2 which are, in fact, defined on S^5.

Finally we apply a similar method to determine T_3. It is again a matter of solving a system of linear equations for the coefficients of the monomials that can comprise T_3. We list the terms involved for the convenience of the reader, who can quite easily check that our claimed T_3 does indeed have the required coproduct. The lead term of this T_3 will play an important role in our subsequent calculations. Indeed, it caused the answer for $v_*(E_7)$ to turn out differently than the author had anticipated.

Momentarily ignoring some homogeneous parts, T_3 must satisfy

$$\begin{aligned} \overline{\psi}(T_3) &= (2vh-3h^2)\otimes(\tfrac{3}{4}h^4-h_2)+(\tfrac{9}{2}h^4-6vh^3+2v^2h^2)\otimes\tfrac{1}{2}h^2 \\ &\quad +(\tfrac{3}{2}h^5-\tfrac{11}{2}vh^4+5v^2h^3-2v^3h^2-vh_2+3h_1h_2-v_2h_1)\otimes(-h). \\ &= -\tfrac{9}{4}h^2\otimes h^4+\tfrac{9}{4}h^4\otimes h^2-\tfrac{3}{2}h^5\otimes h+\tfrac{3}{2}vh\otimes h^4-3vh^3\otimes h^2 \\ &\quad +\tfrac{11}{2}vh^4\otimes h+v^2h^2\otimes h^2-5v^2h^3\otimes h+2v^3h^2\otimes h+3h^2\otimes h_2 \\ &\quad -3hh_2\otimes h-2vh\otimes h_2+vh_2\otimes h+v_2h\otimes h \end{aligned} \tag{5.16}$$

We list the terms that can comprise T_3 along with their coproducts.

$$\begin{aligned}
h^6 &\mapsto 6h\otimes h^5+15h^2\otimes h^4+20h^3\otimes h^3+15h^4\otimes h^2+6h^5\otimes h \\
vh^5 &\mapsto 3h\otimes h^5+5vh\otimes h^4+10vh^2\otimes h^3+10vh^3\otimes h^2+5vh^4\otimes h \\
v^2h^4 &\mapsto -9h^2\otimes h^4+6vh\otimes h^4+4v^2h\otimes h^3+6v^2h^2\otimes h^2+4v^2h^3\otimes h \\
v^3h^3 &\mapsto 27h^3\otimes h^3-27vh^2\otimes h^3+9v^2h\otimes h^3+3v^3h\otimes h^2+3v^3h^2\otimes h \\
v^4h^2 &\mapsto -81h^4\otimes h^2+108vh^3\otimes h^2-54v^2h^2\otimes h^2+12v^3h\otimes h^2+2v^4h\otimes h \\
v^5h &\mapsto 243h^5\otimes h-405vh^4\otimes h+270v^2h^3\otimes h-90v^3h^2\otimes h+15v^4h\otimes h \\
v^2h_2 &\mapsto 3v^2h\otimes h^3+6v^2h^2\otimes h^2+4v^2h^3\otimes h-v^3h\otimes h^2-v^3h^2\otimes h \\
&\quad -9h^2\otimes h_2+6vh\otimes h_2 \\
vhh_2 &\mapsto 3vh\otimes h^4+9vh^2\otimes h^3+10vh^3\otimes h^2+4vh^4\otimes h-v^2h\otimes h^3 \\
&\quad -2v^2h^2\otimes h^2-v^2h^3\otimes h+3h\otimes hh_2+vh\otimes h_2+vh_2\otimes h \\
h^2h_2 &\mapsto 3h\otimes h^5+12h^2\otimes h^4+19h^3\otimes h^3+14h^4\otimes h^2+4h^5\otimes h-vh\otimes h^4 \\
&\quad -3vh^2\otimes h^3-3vh^3\otimes h^2-vh^4\otimes h+2h\otimes hh_2+h^2\otimes h_2 \\
&\quad +2hh_2\otimes h+h_2\otimes h^2 \\
v_2h^2 &\mapsto 24h^4\otimes h^2-35vh^3\otimes h^2+18v^2h^2\otimes h^2-4v^3h\otimes h^2+3h_2\otimes h^2 \\
&\quad +2v_2h\otimes h
\end{aligned}$$

$$\begin{aligned} vv_2h \quad \mapsto \quad & -72h^5 \otimes h + 129vh^4 \otimes h - 89v^2h^3 \otimes h + 30v^3h^2 \otimes h - 4v^4h \otimes h \\ & -9hh_2 \otimes h + 3vh_2 \otimes h + 3v_2h \otimes h \end{aligned}$$

The solution of the resulting system of linear equations is

$$\begin{aligned} (5.17) \quad T_3 \quad = \quad & \tfrac{3}{4}h^6 - \tfrac{1}{2}v^2h^4 + \tfrac{1}{2}v^3h^3 - \tfrac{1}{2}v^2h_2 + vhh_2 - \tfrac{3}{2}h^2h_2 + \tfrac{1}{2}v_2h^2 \\ & +c_1(-\tfrac{81}{2}h^6 + 81vh^5 - \tfrac{135}{2}v^2h^4 + 30v^3h^3 - \tfrac{15}{2}v^4h^2 + v^5h) \\ & +c_2(9h^6 - \tfrac{45}{2}vh^5 + 21v^2h^4 - \tfrac{59}{6}v^3h^3 + 2v^4h^2 + \tfrac{1}{2}v^2h_2 \\ & \quad -3vhh_2 + \tfrac{9}{2}h^2h_2 - \tfrac{3}{2}v_2h^2 + vv_2h). \end{aligned}$$

The first term is rewritten as $\frac{1}{4}(vh^5 - h^5v)$, in order to see it with a unit coefficient. All other terms desuspend to S^9.

The terms T_1, T_5, and T_2 which appear in the equation (5.13) for $\overline{\psi}(T_3)$ which gave rise to the system of equations which we just solved have homogeneous parts whose coefficients we do not know. For example, T_5 includes a summand of $c(-3h^4 + vh^3 + 3h_2)$. Thus added on to the RHS of (5.16) must be $\alpha_2 \otimes c(-3h^4 + vh^3 + 3h^2)$ and 7 other homogeneous parts arising similarly. For each of these we solve a system of equations similar to the one just solved, but with the RHS equal to the appropriate homogeneous term. These give homogeneous summands to T_3. All resulting terms desuspend to S^7, and so may be ignored. We spare the reader the details. □

The terms $\alpha_2 \otimes x_{26}$, $\alpha_2 \otimes x_{18}$, $\alpha_1 \otimes x_{14}$, and $\alpha_1 \otimes x_{10}$ appear in the hypothesis of Proposition 5.11 because of attaching maps in ΩE_7. One might think that care is required as to the coefficients (± 1) of the α_2 and α_1 in Proposition 5.11. However, this is not the case. For if the four terms listed at the beginning of this paragraph are multiplied by units u_1, u_2, u_3, and u_4, respectively, then the terms T_1 to T_6 which are determined in Proposition 5.11 are multiplied by units u_1u_2, $u_1u_2u_3$, $u_1u_2u_3u_4$, u_2u_3, $u_2u_3u_4$, and u_3u_4, respectively. This can be seen by consideration of the first part of the proof of 5.11. For example, we would have $\overline{\psi}(T_1) = u_1\alpha_2 \otimes u_2\alpha_2$.

The terms α_1, α_2, and T_i in 5.11 will be used in the proofs of the theorems throughout the remainder of this section to determine boundary morphisms in exact sequences, and in pulling back terms whose boundary is 0. If units u_i were present as we are discussing here, it will only have the effect of multiplying boundaries and pullbacks by unit amounts. The point is that all terms in a boundary will be multiplied by the same unit, so that cancellation due to different units cannot take place. For example, suppose that a term $h^{I_1}x_{34}$ pulled back to $h^{I_1}x_{34} + h^{I_2}x_{26}$ in the case where all $u_i = 1$. Then, with units u_i present, $h^{I_1}x_{34}$ pulls back to $h^{I_1}x_{34} + u_1h^{I_2}x_{26}$, and the boundary sends this to $h^{I_1} \otimes u_1u_2T_1x_{18} + u_1h^{I_2} \otimes u_2\alpha_2x_{18}$, which is just u_1u_2 times what it would have been. These uniform units do not affect whether terms are zero, and hence can be ignored.

Now we can compute $E_2^{s,2j}(Y_7)$, dividing into cases depending upon the parity and mod 9 value of j. These will be delineated in Theorems 5.18, 5.23, 5.29, 5.32, 5.34, and 6.1. Note that the exact sequences in E_2 induced by (5.8), (5.9), and (5.10), together with (5.19), imply that if t is even, then $E_2^{s,t}(Y_7) = 0$ unless $s = 1$ or 2.

The first case is as follows.

Theorem 5.18. *If j is odd, and $j \equiv 1$ or 7 mod 9, then*

$$E_2^{1,2j}(Y_7) \approx E_2^{2,2j}(Y_7) \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^{\min(8,\nu(j-43)+5)}.$$

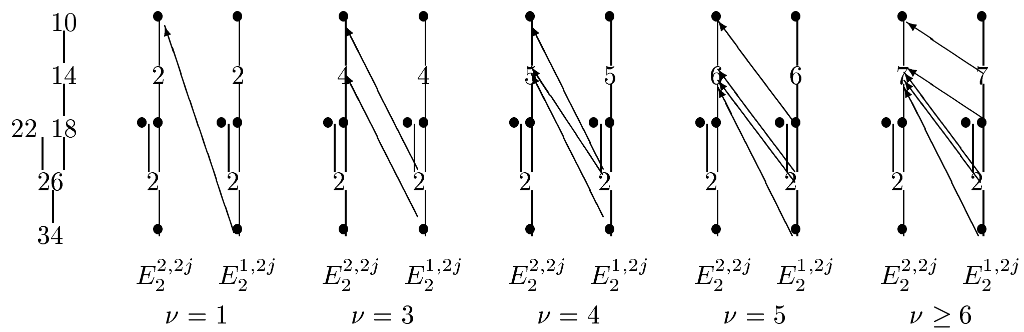
Proof. Let j be as in the theorem, and $\nu = \nu(j-7)$. Formally, we obtain the result by computing first the exact sequence in $E_2(-)$ associated to (5.8), then that associated to (5.9), and then that associated to (5.10). We know from (2.5) and 2.15 that

$$(5.19) \qquad E_2^{s,2j}(2m) \approx \begin{cases} \mathbf{Z}/3^{\min(m,\nu(j-m)+1)} & \text{if } j \equiv m \bmod 2, \text{ and } s = 1 \text{ or } 2 \\ 0, & \text{otherwise} \end{cases}$$

and we know from [13, 2.4] how to compute $E_2(10,14)$ from $E_2(10)$ and $E_2(14)$. These are the building blocks, but the glue is the boundary morphisms in the exact sequences, and computing these requires much care.

A convenient way to picture the calculations is by Diagram 5.20, which we think of as resembling an Adams spectral sequence chart.

Diagram 5.20.



Each $\bullet$ represents a $\mathbf{Z}/3$, and each integer e represents a $\mathbf{Z}/3^e$. These groups correspond to $E_2(2m)$, where $2m$ is the integer indicated on the left side of the diagram. The vertical lines indicate nontrivial extensions (multiplication by 3). These are true because of the α_1 and α_2 attaching maps and Theorem 3.1. The positioning of the 22-class is due to (5.5), i.e., that it is split away from the 10-, 14-, and 18-classes.

For example, the diagram for the case $\nu = 3$ means that if $\nu(j-7) = 3$ the boundary morphisms in (5.8) and (5.9) are 0, yielding $E_2^{1,2j}(18,26,34) \approx E_2^{2,2j}(18,26,34) \approx \mathbf{Z}/3^4$, while in (5.10)

$$E_2^{1,2j}(18,26,34) \xrightarrow{\partial} E_2^{2,2j}(22) \oplus E_2^{2,2j}(10,14) \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^5$$

is 0 into the first summand and has image of order 3^2 in the second summand. Using either the exact sequence (5.10) or the diagram, this implies that in this case

$$E_2^{1,2j}(Y_7) \approx E_2^{2,2j}(Y_7) \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^7.$$

The case $\nu = 2$ omitted from Diagram 5.20 has groups of order 3^3, i.e. labeled "3," on the 14-cell, and otherwise has the same groups as do the other values of ν. If $(j-7)/18 \equiv 1$ mod 3, then it has a differential like that in the case $\nu = 1$, while if $(j-7)/18 \equiv 2$ mod 3, then it has no nonzero differentials.

Of course, we still have to verify that the differentials are as claimed in Diagram 5.20 and the above paragraph describing the case $\nu = 2$. The reader can easily verify that this will imply Theorem 5.18. The $\mathbf{Z}/3$ on the 22-cell splits for algebraic reasons.

We analyze the differentials by the methods used extensively in [23] and [20], involving the unstable cobar complex. One convention is that we often omit writing powers of v_1 on the left; they can always be determined by consideration of total degree. The boundary $E_2^{1,2j}(34) \xrightarrow{\partial} E_2^{2,2j}(26)$ sends the generator $h\iota_{34}$ to $h \otimes \alpha_2\iota_{26} = h \otimes (hv + vh)\iota_{26}$. This is obtained from $\psi(x_{34}) = 1 \otimes x_{34} + \alpha_2 \otimes x_{26}$ in 5.11, and from 2.19. The relationship of ∂ with the coaction is standard; see, e.g., [20, 2.7].

We use 2.20 to write $h \otimes vh = vh \otimes h - 3h^2 \otimes h$, and $3h^2 \otimes h = h^2 \otimes (v - \eta v) = 0$. Also, $h \otimes hv$ is defined on S^1 and hence is 0 in $E_2(-)$. So the image of ∂ equals $v^{\mathrm{pwr}} h \otimes h\iota_{26}$. By 2.18(second part of (2)), this cycle equals $d((h^3 + L)\iota_{26})$, with $\nu = \nu(j - 13) = 1$ in 2.18. Here we have omitted a unit coefficient, which will be done routinely unless the coefficient plays a significant role. Here L desuspends **lower** than the associated term, in this case h^3, a notation that will be employed frequently, with the L's sometimes adorned with primes to distinguish them from one another.

Thus the generator $h\iota_{34}$ pulls back to $h\iota_{34} + (h^3 + L)\iota_{26}$ in $E_2^{1,2j}(26, 34)$, and ∂ in (5.9) sends this to $(h \otimes (\frac{1}{2}v^2h^2 + L') + (h^3 + L) \otimes \alpha_2)\iota_{18} \in E_2^{2,2j}(18)$. Here we have used 5.11. The leading term here is

$$\tfrac{1}{2}h \otimes v^2h^2 = \tfrac{1}{2}(\eta v)^2 h \otimes h^2 = \tfrac{1}{2}(v - 3h)^2 h \otimes h^2,$$

which has leading term $\frac{1}{2}h \otimes h^2$. Note how v's on the left are absorbed into other unstated v's. By 2.18(2), this equals $d((h^3 + L'')\iota_{18})$ (omitting unit coefficients), and so our generator pulls back to

$$z \equiv h\iota_{34} + h^3\iota_{26} + h^3\iota_{18} \in E_2^{1,2j}(18, 26, 34).$$

Here, and subsequently, "$\equiv$" will mean "mod L," with the lower terms varying from term to term.

We analyze the two components of $\partial(z)$ in the exact sequence of E_2 derived from (5.10). We begin by showing that the component ∂_2 into $E_2^{2,2j}(22)$ is 0. We have $\partial_2(z) = (h \otimes (h^3 + L) + h^3 \otimes h)\iota_{22}$. Here we use the α_1 attaching map from 22 to 26 in ΩE_7, which causes the $\otimes h\iota_{22}$. The $\otimes(h^3 + L)$ is obtained by the same calculation that gave T_4 in Proposition 5.11. But these terms don't even matter very much, for such terms desuspend far below S^{22}, and hence are 0 in $E_2^{2,2j}(22) \approx \mathbf{Z}/3$. Here we use a fact that we will use frequently, essentially from 2.15(5), that if $E_2^{2,2j}(2n + \epsilon) \approx \mathbf{Z}/p$, then an element in it which is in the image of the double desuspension is 0.

Similar, although much more delicate, considerations apply to obtaining the other component

$$\partial_1 : E_2^{1,2j}(18, 26, 34) \to E_2^{2,2j}(10, 14). \tag{5.21}$$

First we determine the composite when ∂_1 is followed (by ρ) into $E_2^{2,2j}(14)$. Using Proposition 5.11 and the usual relationship between the coaction and the boundary morphism, we obtain

$$\rho\partial_1(z) \equiv (h \otimes vh^4 + h^3 \otimes h^3 + h^3 \otimes h)\iota_{14}. \tag{5.22}$$

Here all terms except $h \otimes vh^4$ desuspend to S^7, while, mod S^7, $h \otimes vh^4 \equiv h \otimes h^4$. If $\nu \geq 3$, then, by 2.18(1), this element has order $3^{\min(4,\nu-2)}$ in $E_2^{2,2j}(14)$, and this is as claimed in Diagram 5.20, with the arrows above the lowest one being a consequence of the lowest one and the extensions.

If $\nu = 1$, then by 2.18(2) and (5.22), $\rho\partial_1(z) = d((h^6 + L)\iota_{14})$. Thus z pulls back to

$$z' \equiv h\iota_{34} + h^3\iota_{26} + h^3\iota_{18} + h^6\iota_{14} \in C(14, 18, 26, 34).$$

Using 5.11, this satisfies

$$\partial(z') \equiv (h \otimes vh^5 + h^3 \otimes vh^3 + h^3 \otimes h^2 + h^6 \otimes h)\iota_{10}.$$

Here there can be "lower" terms associated with the factor on either side of the tensor sign, omitted v's occur only on the left, and, as usual, unit coefficients are omitted. All terms here except the first desuspend to S^9, while that term generates $E_2^{2,2j}(10)$, so the image of ∂_1 in this case has order 3 in $E_2^{2,2j}(10, 14)$, as claimed.

Finally we consider the delicate case when $\nu = 2$. In this case, there are two terms with the potential to cancel, and so we must keep track of unit coefficients. We write $j = 7 + 18c$, with $c \not\equiv 0$ mod 3. As before, $E_2^{1,2j}(18, 26, 34)$ is generated by $z \equiv h\iota_{34} + h^3\iota_{26} + h^3\iota_{18}$. The unit coefficients of the second and third terms will not be important, and so are omitted. The leading term of $\rho\partial_1(z)$ in $E_2^{2,2j}(14)$ is, by 5.11, $h \otimes (-5vh^4)\iota_{14} \equiv h \otimes h^4\iota_{14}$, where we have used that $-5 \equiv 1$ mod 3. By Lemma 2.21(3), $d(h^7) \equiv -9ch \otimes h^6$ in this stem. (($\ell + n + 1$) of the lemma multiplied by $2(p-1)$ equals $2j - 14$.) Thus, since $9h^6 \equiv h^4$, we obtain $\rho\partial_1(z) = d((-\frac{1}{c}h^7 + L)\iota_{14})$, and so z pulls back to

$$z' \equiv h\iota_{34} + h^3\iota_{26} + h^3\iota_{18} + \tfrac{1}{c}h^7\iota_{14}.$$

This satisfies

$$\partial(z') \equiv (h \otimes \tfrac{1}{4}vh^5 + uh^3 \otimes vh^3 + u'h^3 \otimes h^2 + \tfrac{1}{c}h^7 \otimes (-h))\iota_{10}$$

with u and u' units in $\mathbf{Z}_{(3)}$. The middle terms desuspend, while the first and last combine, using 2.21(2), to give $\frac{1}{4} + \frac{1}{c}$ times the generator of $E_2^{2,2j}(10) \approx \mathbf{Z}/3$. This is nonzero if $c \equiv 1$ mod 3, and 0 if $c \equiv 2$ mod 3, as claimed in the paragraph earlier in the proof which described the case $\nu = 2$. □

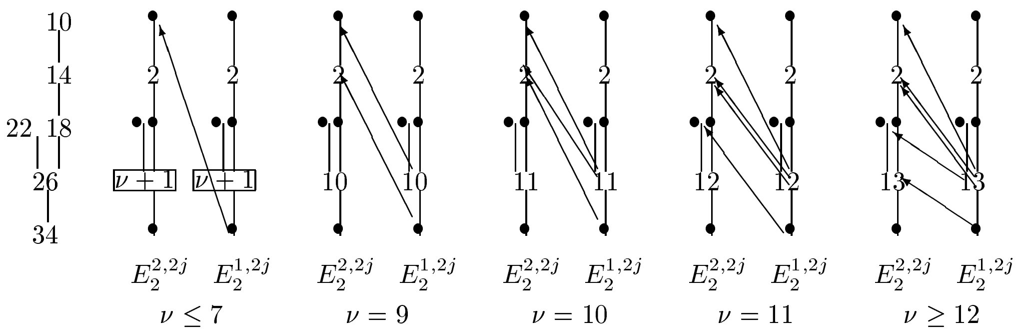
The statement and proof for the case $j \equiv 4$ mod 9 are quite similar to the cases just completed.

Theorem 5.23. *If j is odd, and $j \equiv 4$ mod 9, then*

$$E_2^{1,2j}(Y_7) \approx E_2^{2,2j}(Y_7) \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^{\min(14,\nu(j-13-4\cdot3^8)+5)}.$$

Proof. Let j be as in the theorem, and $\nu = \nu(j-13)$. As in the previous theorem, the way in which the result stated in the theorem is obtained is most conveniently expressed in a diagram.

Diagram 5.24.



The omitted case $\nu = 8$ is like the case $\nu \le 7$ if $(j-13)/(2\cdot 3^8) \equiv 1$ mod 3, while it has all differentials 0 if $(j-13)/(2\cdot 3^8) \equiv 2$ mod 3. In most cases, the $\mathbf{Z}/3$ from the 22-class splits for algebraic reasons. The splitting in the cases when $\nu \ge 11$ require a bit of care, which will be dealt with later in the proof. The boundary in the diagram in these cases is meant to be hitting the sum of the classes on the 22 and the 18.

We begin with the case $\nu \le 7$. We start as in the proof of 5.18, but this time the boundary of $h\iota_{34}$ in $E_2^{2,2j}(26)$ is $d((h^{\nu+2}+L')\iota_{26})$, by 2.18(2). (In the proof of 5.18, we had $\nu(j-13)=1$.) Thus the generator of $E_2^{1,2j}(26,34)$ equals, mod lower terms, $h\iota_{34} + h^{\nu+2}\iota_{26}$. The next term is found by writing

$$(5.25)\qquad (h\otimes(\tfrac{1}{2}v^2h^2+L)+(h^{\nu+2}+L')\otimes(2vh-3h^2))\iota_{18}$$

as a boundary in the unstable cobar complex. The first term will dominate if $\nu \le 4$, while the second term will dominate if $4 \le \nu \le 7$. (If $\nu = 4$, there could be cancellation that would cause it to desuspend even lower, but that won't affect the final result.) We obtain that

$$z \equiv h\iota_{34} + h^{\nu+2}\iota_{26} + h^{\max(3,\nu-1)}\iota_{18}$$

generates $E_2^{1,2j}(18,26,34)$. The $h^{\nu-1}\iota_{18}$ when $\nu \ge 4$ is obtained since

$$h^{\nu+2}\otimes vh \equiv h^{\nu}\otimes h \equiv h\otimes h^{\nu-2} = d(h^{\nu-1}),$$

using 2.21(1,2,3). Now

$$\rho\partial_1(z) \equiv (h\otimes vh^4 + h^{\nu+2}\otimes h^3 + h^{\max(3,\nu-1)}\otimes h)\iota_{14},$$

which has leading term a multiple, k, of $h\otimes h^4$. By 2.18(2) this is $d(kh^6\iota_{14})$ since $\nu(2j-14)=1$, and so z pulls back to $z' \equiv z - kh^6\iota_{14}$ in $C(14,18,26,34)$. The leading term of $\partial(z')$ in $E_2^{2,2j}(10)$ is $h\otimes vh^5$, which is a generator. It is also important to know here that $E_2^{1,2j}(18,26,34)\xrightarrow{\partial}E_2^{2,2j}(22)$ is 0, for if it were nonzero then $E_2^{2,2j}(Y_7)$ would be cyclic. The leading term of this ∂ is $h^{\nu+2}\otimes h\iota_{22}$ which desuspends and hence is 0 in E_2.

Next we consider the case $\nu = 8$, in which we have to keep track of unit coefficients because of the possibility of two cancelling terms. Let $j-13 = 2\cdot 3^8 c$, with $c \not\equiv 0$ mod 3. By 2.21(3), we have $d(h^{10}\iota_{26}) \equiv -3^8 ch\otimes h^9\iota_{26} \equiv -ch\otimes h\iota_{26}$, where the second step utilizes $3^8h^8 = (v-\eta v)^8$. Then $E_2^{1,2j}(34)\xrightarrow{\partial}E_2^{2,2j}(26)$ sends the generator to $h\otimes(-h)\iota_{26} = d((\frac{1}{c}h^{10}+L)\iota_{26})$, and so the generator pulls back to

$z \equiv h\iota_{34} - \frac{1}{c}h^{10}\iota_{26}$. The leading term of $\partial(z)$ in $E_2^{2,2j}(18)$ is $-\frac{1}{c}h^{10}\otimes 2vh\iota_{18}$, which, using 2.21(1,2), is equivalent to $-\frac{2}{c}h^8 \otimes h\iota_{18} \equiv \frac{2}{c}h \otimes h^6\iota_{18}$. By 2.21(3),

$$d(h^7\iota_{18}) \equiv -\tfrac{1}{2}(j-9)h \otimes h^6\iota_{18} = -(2+3^8c)h \otimes h^6\iota_{18} \equiv -c\partial(z).$$

Thus z pulls back to $z' \equiv z + \frac{1}{c}h^7\iota_{18}$. The leading term of $\rho\partial_1(z')$ in $E_2^{2,2j}(14)$ is $\frac{1}{c}h^7 \otimes (-h)\iota_{14}$. By 2.21(3) again,

$$d(h^7\iota_{14}) \equiv -\tfrac{1}{2}(j-7)h \otimes h^6\iota_{14} \equiv -h \otimes 3h^6\iota_{14} \equiv -h \otimes h^5\iota_{14} \equiv -c\partial(z'),$$

where we have used 2.21(1) at the last step. Thus z' pulls back to $z'' \equiv z' + \frac{1}{c}h^7\iota_{14}$ in $C(14,18,26,34)$. There are two leading terms in $\partial(z'') \in E_2^{2,2j}(10)$. These are $h \otimes \frac{1}{4}vh^5$ and $\frac{1}{c}h^7 \otimes (-h)$. They combine to give $\frac{1}{4}+\frac{1}{c}$ times a generator, and this is 0 if $c \equiv 2 \bmod 3$, and nonzero if $c \equiv 1 \bmod 3$, as claimed. The ∂ into the 22-part is 0 as in the case $\nu \le 7$.

If $9 \le \nu \le 10$, the situation is much easier. Similarly to the previous cases, but ignoring units, the generator of $E_2^{1,2j}(18,26,34)$ is $z \equiv h\iota_{34} + h^{\nu+2}\iota_{26} + h^{\nu-1}\iota_{18}$. The leading term of $\rho\partial_1(z)$ in $E_2^{2,2j}(14)$ is $h^{\nu-1} \otimes h\iota_{14}$, which is a generator if $\nu = 10$, and is 3 times the generator if $\nu = 9$. The boundary $E_2^{1,2j}(18,26,34) \xrightarrow{\partial_2} E_2^{2,2j}(22)$ is 0 because its leading term is $h^{\nu+2} \otimes h\iota_{22}$ which is 0 in E_2 for $\nu \le 10$.

When $\nu = 11$, the boundary from $E_2^{1,2j}(26,34)$ to $E_2^{2,2j}(18)$ is now nonzero. Indeed, its image, given in (5.25), has leading term

$$h^{13} \otimes 2vh\iota_{18} \equiv 2h^{11} \otimes h\iota_{18} \equiv h \otimes h^9\iota_{18},$$

which is a generator. Here we have used 2.21(1) and 2.21(2). The boundary from $E_2^{1,2j}(26,34)$ to $E_2^{2,2j}(22)$ is also nonzero since the generator $z \equiv h\iota_{34} + h^{13}\iota_{26}$ satisfies $\partial(z) \equiv h^{13} \otimes h\iota_{22}$, and this is a generator. The chart would then suggest (accurately) that the boundary hits into the sum of the two classes, and the extension is also into this sum. One way to formalize this uses the exact sequence

$$E_2^{1,2j}(26,34) \xrightarrow{\partial} E_2^{2,2j}(10,14,18,22) \to E_2^{2,2j}(Y_7) \to E_2^{2,2j}(26,34). \tag{5.26}$$

The first and last groups are $\mathbf{Z}/3^{13}$, while the second is $\mathbf{Z}/3 \oplus \mathbf{Z}/3^4$. The boundary ∂ hits the sum of the two generators. There is a cycle representative z in $E_2^{2,2j}(Y_7)$ which projects to an element of order 3 in $E_2^{2,2j}(26,34)$ and satisfies that 3 times this generator is the image of the sum of the two generators of $E_2^{2,2j}(10,14,18,22)$. This implies $E_2^2(Y_7) \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^{13}$.

Actually, a little bit more care is required here with regard to coefficients of the generators. It is conceivable that the boundary could hit the sum of generators but the extension be into their difference, and then the extension group would be cyclic of order 3^{14}. What really happens is that, if c is defined as before by $j-13 = 2\cdot 3^{11}c$, then a generator $z \equiv h\iota_{34} - \frac{1}{c}h^{13}\iota_{26}$ satisfies

$$\partial(z) \equiv \tfrac{1}{c}h^{13} \otimes h\iota_{22} - \tfrac{1}{c}h^{13} \otimes 2vh\iota_{18}, \tag{5.27}$$

while on the other hand, the argument of Theorem 3.1 shows that the element $d(3^{j-27}h^{j-13})\iota_{26}$ of order 3 extends to a cycle z' in $C(18,22,26)$ such that, mod classes that desuspend farther, $3z'$ is homologous to

$$3^{j-26}h^{j-13} \otimes (-h)\iota_{22} + 3^{j-26}h^{j-13} \otimes 2vh\iota_{18}. \tag{5.28}$$

The classes in (5.27) and (5.28) are clearly unit multiples of one another. In each case, we use $h^{13} \otimes vh \equiv h^{11} \otimes h$ to see that the second term is a generator.

This completes the case $\nu = 11$. The case $\nu \geq 12$ is very similar. Actually it is a bit easier, for the consideration of the previous paragraph need not be addressed, since the initial differential hits into a cyclic group. □

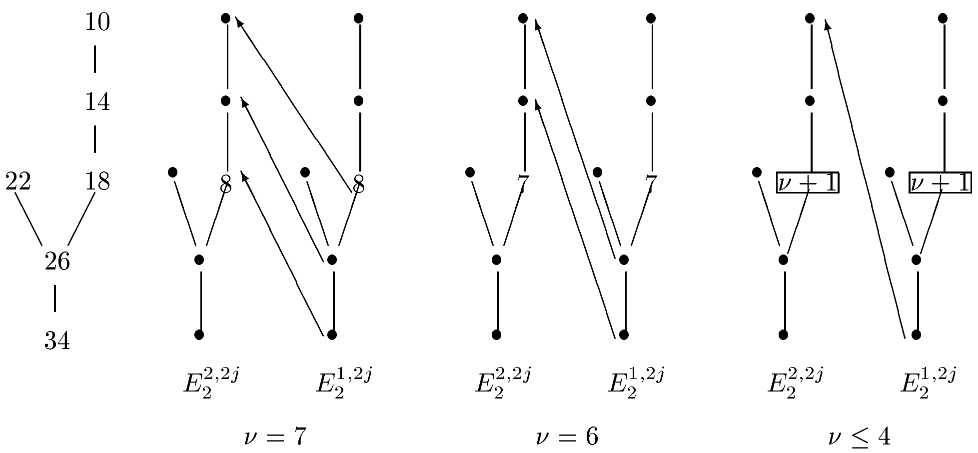
The case $j \equiv 0 \bmod 3$ introduces no new ideas.

Theorem 5.29. *If j is odd, and $j \equiv 0 \bmod 3$, then*

$$E_2^{1,2j}(Y_7) \approx E_2^{2,2j}(Y_7) \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^{\min(10,\nu(j-9-2\cdot 3^5)+4)}.$$

Proof. Let j be as in the theorem, and $\nu = \nu(j-9)$. We will show that the diagram encapsulating the exact sequences of (5.8), (5.9), and (5.10) is as depicted in Diagram 5.30 for certain values of ν. This diagram, together with the subsequent discussion of what happens for values of ν not included in the diagram, implies Theorem 5.29.

Diagram 5.30.



If $\nu \geq 8$, then the group corresponding to the 18-cell has order 3^9, and by [13, 2.4] there is a nonzero boundary morphism from $E_2^{1,2j}(26)$ to $E_2^{2,2j}(18)$ (hitting the element of order 3, of course), and three other boundary morphisms (one below and two above it) follow from it by the extensions in a diagram similar to that of 5.30.

We will show below that the case $\nu = 5$ is like the case $\nu \leq 4$ if $(j-9)/(2\cdot 3^5) \equiv 2 \bmod 3$, while it has no differentials if $(j-9)/(2\cdot 3^5) \equiv 1 \bmod 3$. But first we establish that the cases in Diagram 5.30 are as depicted.

Let $\nu = 7$. The nonzero differential from $E_2^{1,2j}(34) \approx \mathbf{Z}/3$ to $E_2^{2,2j}(18) \approx \mathbf{Z}/3^8$ is established similarly to that in the case $\nu = 11$ in the preceding theorem. Indeed, the boundary $E_2^{1,2j}(34) \xrightarrow{\partial} E_2^{2,2j}(26)$ sends the generator to

$$h \otimes \alpha_2 \iota_{26} = h \otimes (2vh - 3h^2)\iota_{26} = (2vh \otimes h - 6h^2 \otimes h - 3h \otimes h^2)\iota_{26} = d(vh^2 - 2h^3)\iota_{26},$$

and so it pulls back to $h\iota_{34} + (2h^3 - vh^2)\iota_{26}$. This in turn has boundary $(h \otimes (\frac{1}{2}v^2h^2 + L) + (2h^3 - vh^2) \otimes \alpha_2)\iota_{18}$, whose leading term $\frac{1}{2}h \otimes v^2h^2\iota_{18} \equiv \frac{1}{2}h \otimes h^2\iota_{18}$ has order 3 by Theorem 2.18(1). The two differentials above this differential then follow from the extensions. They could also be obtained by the method of pulling back cycles that we have been using.

When $\nu \le 6$, the generator of $E_2^{1,2j}(18,26,34)$ is $z \equiv h\iota_{34} + h^3\iota_{26} + h^{\nu+3}\iota_{18}$. To obtain the last term, we used 2.21(3) to write $h \otimes h^2\iota_{18}$ as $d(h^{\nu+3}\iota_{18})$ mod lower terms. The leading term of $\rho\partial_1(z)$ in $E_2^{2,2j}(14)$ is $h^{\nu+3} \otimes h\iota_{14}$, which is a generator if $\nu = 6$. If $\nu < 6$, then this is $d((h^{\nu+2} + L)\iota_{14})$, and so z pulls back to $z' \equiv z + h^{\nu+2}\iota_{14}$. Then $\partial(z')$ in $E_2^{2,2j}(10)$ is

$$(h \otimes vh^5 + h^3 \otimes vh^3 + h^{\nu+3} \otimes h^2 + h^{\nu+2} \otimes h)\iota_{10}, \tag{5.31}$$

using Proposition 5.11. If $\nu < 5$, then the first term is the leading term, and it is a generator. If $\nu = 5$, we must keep track of unit coefficients, since the first and last terms have the same excess.

Let $j - 9 = 2 \cdot 3^5 c$, with $c \not\equiv 0$ mod 3. We start with $h\iota_{34}$. The next term $(h^3\iota_{26})$ is insignificant. The leading term of the image under $\xrightarrow{\partial} E_2^{2,2j}(18)$ is $h \otimes \frac{1}{2}v^2h^2 \equiv \frac{1}{2}h \otimes h^2$. Incorporating coefficients into the analysis of the previous paragraph, 2.21(3) actually says that $d(h^8\iota_{18}) \equiv -3^5ch \otimes h^7\iota_{18} \equiv -ch \otimes h^2\iota_{18}$, and so z is actually equivalent to $h\iota_{34} + h^3\iota_{26} + \frac{1}{2c}h^8\iota_{18}$. The leading term of $\rho\partial_1(z)$ is

$$\tfrac{1}{2c}h^8 \otimes (-h)\iota_{14} \equiv \tfrac{1}{2c}h \otimes h^6\iota_{14} \equiv -d(\tfrac{1}{2c}h^7\iota_{14}),$$

since by 2.21(3) $d(h^7\iota_{14}) \equiv -\frac{1}{2}(2 \cdot 3^5 + 2)h \otimes h^6\iota_{14}$. Thus the refined form of z' has significant terms $h\iota_{34} + \frac{1}{2c}h^7\iota_{14}$, and so the leading terms of $\partial(z')$ are

$$(h \otimes \tfrac{1}{4}vh^5 + \tfrac{1}{2c}h^7 \otimes (-h))\iota_{10} \equiv (\tfrac{1}{4}h \otimes h^5 + \tfrac{1}{2c}h \otimes h^5)\iota_{10},$$

and this is 0 in E_2 if $c \equiv 1$ mod 3, and is a generator if $c \equiv 2$ mod 3.

The boundary into $E_2^{2,2j}(22)$ is $(h \otimes h^3 + h^3 \otimes h)\iota_{22}$, which is 0 when the group is isomorphic to $\mathbf{Z}/3$. □

The next result also follows by the methods already employed. Note however the excluded case, which requires major refinements, deferred to the next section.

Theorem 5.32. *If j is odd, and $j \equiv 5$ or 8 mod 9, but $\nu(j-17) \ne 13$, then*

$$E_2^{1,2j}(Y_7) \approx E_2^{2,2j}(Y_7) \approx \mathbf{Z}/3^2 \oplus \mathbf{Z}/3^{\min(17,\nu(j-17)+4)}.$$

Proof. The proof when $j \equiv 5$ is particularly simple. The result here is just that $E_2^{1,2j}(Y_7) \approx E_2^{2,2j}(Y_7) \approx \mathbf{Z}/3^2 \oplus \mathbf{Z}/3^5$. It is most conveniently seen with charts such as those of the earlier proofs in this section. In this case, the two main towers have groups of exponent 2, 1, 1, 1, and $\min(\nu(j-5)+1, 5)$, reading from bottom to top. These are the groups corresponding to generators of dimensions 34, 26, 18, 14, and 10, respectively. There is also a group of exponent 2 from the 22-class, and it extends cyclically above the lowest 1.

We will show that the boundary is nonzero from enough of the bottom groups of the $E_2^{1,2j}$-tower to just kill the group of exponent $\min(\nu(j-5)+1, 5)$ at the top of the $E_2^{2,2j}$-tower. That leaves a $\mathbf{Z}/3^5$ in each tower, and the $\mathbf{Z}/3^2$ coming from the 22-class cannot be involved in differentials and must split off for algebraic reasons.

Let $\nu = \nu(j-5)$. To see these boundary morphisms, we show that the element at the top of the $E_2^{2,2j}$-tower (i.e., the element of order 3 in $E_2^{2,2j}(10)$) is hit by the $\mathbf{Z}/3$ on 14 if $\nu \ge 4$, by the $\mathbf{Z}/3$ on 18 if $\nu = 3$, and by the $\mathbf{Z}/3$ on 26 if $\nu = 2$. Other differentials are seen from the cyclic extensions by reading down the towers. The differential when $\nu \ge 4$ was proved in [13, 2.4]. The differential when $\nu = 3$ is seen by pulling the generator of $E_2^{1,2j}(18)$ back to $z \equiv h\iota_{18} + h^2\iota_{14}$ and then using 5.11 to obtain $\partial(z) \equiv h \otimes \frac{1}{2}v^2h^2\iota_{10}$. The v^2 can be moved to the left using 2.20(1), and by 2.18(1) $\frac{1}{2}h\otimes h^2\iota_{10}$ is an element of order 3 in $E_2^{2,2j}(10)\approx \mathbf{Z}/3^4$. The case $\nu=2$ is similar, with the leading term of $\partial(h\iota_{26}+h^2\iota_{18}+h^4\iota_{14})$ being $h\otimes vh^3\iota_{10}$, which has order 3 in $E_2^{2,2j}(10)\approx\mathbf{Z}/3^3$. This completes the proof when $j\equiv 5$ mod 9.

Now suppose $j\equiv 8$ mod 9, and let $\nu=\nu(j-17)$. The picture is similar to that just described, with groups of exponent $\min(17,\nu+1)$, 1, 1, 1, and 2, from bottom to top, and a group of exponent 2 extending just above the lowest 1. These groups correspond to generators of dimensions 34, 26, 18, 14, 10, and 22, as in the case $j\equiv 5$ just considered. The claim is that differentials from the $E_2^{1,2j}$-tower kill all but the bottom 3^{17} elements in the $E_2^{2,2j}$-tower if $\nu\geq 14$, and that they kill the top $\mathbf{Z}/3^2$ if $\nu\leq 12$. Actually, when $\nu\geq 14$, the initial element hit also involves a summand in the 22-summand, but these elements hit are just the appropriate 3-power times the element at the bottom of the $E_2^{2,2j}$-tower. The $E_2^{2,2j}(22)\approx\mathbf{Z}/3^2$ is a split summand in $E_2^{2,2j}(Y_7)$ even though it may be a summand of a class hit by a boundary. We will illustrate this carefully in the case $\nu=15$ below.

When $\nu\geq 16$, the differential from the bottom of the tower into the class on the 26-class follows from [13, 2.4]. Of course, the remaining differentials in this case follow from the extensions.

When $\nu=15$, the generator of $E_2^{1,2j}(34)$ has leading term $h^{16}\iota_{34}$ by 2.15(2), and this pulls back to $z\equiv h^{16}\iota_{34}+uh^{13}\iota_{26}$, where u is a unit in $\mathbf{Z}_{(3)}$. Usually we don't bother to list these unit coefficients, and here the value of u will not be important, but because cancellation issues will come into play, we feel that the unit should at least be given lip service. The leading term of $\partial(z)$ in $E_2^{2,2j}(18)\oplus E_2^{2,2j}(22)$ is

$$uh^{13}\otimes 2vh\iota_{18}+uh^{13}\otimes(-h)\iota_{22}. \tag{5.33}$$

Using 2.21, each of these terms is a generator of its summand. On the other hand, as in the case $\nu=11$ of the proof of 5.23, there is a cycle z' in $C(18,22,26)$ which restricts to a generator of $E_2^{2,2j}(26)$, and has $3z'$ homologous to a unit times (5.33). To clarify the splitting, that $E_2^{2,2j}(22)\approx\mathbf{Z}/3^2$ splits as a direct summand of $E_2^{2,2j}(Y_7)$, we again use the exact sequence (5.26). The argument following (5.26) applies verbatim, with $\mathbf{Z}/3^{13}$ and $\mathbf{Z}/3$ replaced by $\mathbf{Z}/3^{17}$ and $\mathbf{Z}/3^2$, respectively.

The case $\nu=14$ is similar, but involves a 2-step extension process. In the diagram of the type 5.30, $E_2^{2,2j}(26)$ extends into $E_2^{2,2j}(22)\approx\mathbf{Z}/3^2$ and into a $\mathbf{Z}/3^2$ built from $E_2^{2,2j}(18)$ and $E_2^{2,2j}(14)$.[1] The boundary hits into an element of order 3 in each of these summands, which in the case of the second summand means that it hits a generator of $E_2^{2,2j}(14)$. In order to know that the splitting is as claimed, we must verify that the element hit is 3^2 times a generator of $E_2^{2,2j}(26)$. This is the same sort of verification that we have been making in some other cases, i.e. that the boundary and the extension involve classes that are unit multiples of one another, but here the extension is a 2-step process.

Boundary: The generator of $E_2^{1,2j}(34)$ pulls back to $z\equiv h^{15}\iota_{34}+uh^{12}\iota_{26}$, with u a unit. The component of the boundary of this in $E_2^{2,2j}(22)$ is $uh^{12}\otimes(-h)\iota_{22}$. On the other hand, the boundary into $E_2^{2,2j}(18)$ satisfies

$$\partial(z)\equiv uh^{12}\otimes vh\iota_{18}\equiv 2uh^{10}\otimes h\iota_{18}\equiv -2uh\otimes h^8\iota_{18}=d(2uh^9\iota_{18}).$$

[1] $E_2^{2,2j}(10)$ is in the image of ∂, and hence does not figure into the extension question being considered here.

Here we have used the three parts of 2.21, with the last step using that $2j-18=2(2\cdot 3^{14}c+8)$, and so $\frac{1}{4}(2j-18)\equiv 1 \bmod 3$. Thus z pulls back to $z'\equiv z-2uh^9\iota_{18}$, and the leading term of $\partial(z')$ is $2uh^9\otimes h\iota_{14}$.

Extension: Similarly to (5.28), $d(h^{14})\iota_{26}$ is an element of order 3 in $E_2^{2,2j}(26)$, and it extends to a cycle z' in $C(18,22,26)$ such that, mod lower classes, $3z'$ is homologous to

$$h^{13}\otimes(-h)\iota_{22}+h^{13}\otimes 2vh\iota_{18}.$$

To evaluate $3^2z'$, we use the second 3 to reduce each h^{13} to h^{12}. The second term becomes

$$2h^{10}\otimes h\iota_{18}\equiv -2h\otimes h^8\iota_{18}\equiv d(2h^9)\iota_{18}\equiv -2h^9\otimes(-h)\iota_{14}.$$

Here we have applied (3.2) at the last step.

Thus we have a unit times $h^{12}\otimes(-h)\iota_{22}+2h^9\otimes h\iota_{14}$ as the leading term of both the image of the boundary, and the 3^2-multiple of the generator.

The case $\nu\le 12$ is much easier. The generator of $E_2^{1,2j}(34)$ pulls back to $z\equiv h^{\nu+1}\iota_{34}+h^{\nu-2}\iota_{26}+h^{\nu-5}\iota_{18}+h^{\nu-6}\iota_{14}$ and this satisfies $\partial(z)\equiv h^{\nu+1}\otimes vh^5\iota_{10}$, which is a generator since it does not desuspend. $\square$

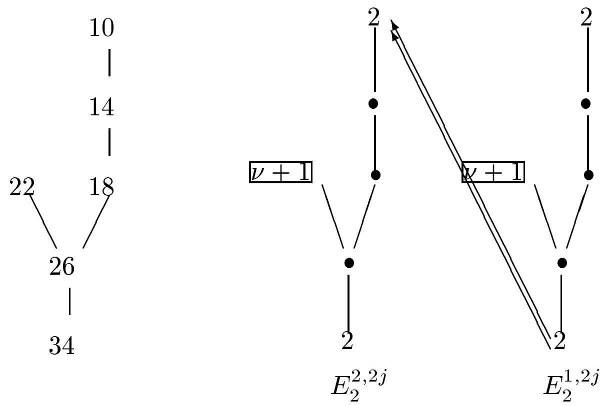
The final case differs from the others in that $E_2^{1,2j}(Y_7)$ and $E_2^{2,2j}(Y_7)$ are not isomorphic.

Theorem 5.34. *Assume j is odd and $j\equiv 2$ mod 9. Then* $E_2^{2,2j}(Y_7)\approx \mathbf{Z}/3^2\oplus\mathbf{Z}/3^{\min(13,\nu(j-11)+4)}$, *while*

$$E_2^{1,2j}(Y_7)\approx\begin{cases}\mathbf{Z}/3^3\oplus\mathbf{Z}/3^5 & \text{if } \nu(j-11)=2\\ \mathbf{Z}/3^4\oplus\mathbf{Z}/3^{\min(11,\nu(j-11)+2)} & \text{if } \nu(j-11)>2.\end{cases}$$

Proof. Let j be as in the theorem, and $\nu=\nu(j-11)$. The picture when $\nu<8$ is as in Diagram 5.35.

Diagram 5.35.



$\nu<8$

The indicated boundary is seen by pulling back the generator $\alpha_{m/2}\iota_{34}$ to a cycle z on $C(14,18,22,26,34)$, and then obtaining $\alpha_{m/2}\otimes vh^5\iota_{10}$ as the leading term of

$\partial(z)$. This generates $E_2^{2,2j}(10)$. This generator $\alpha_{m/2}$ is as described in 2.15(1). The slash does not mean division; this notation was introduced in papers preceding [8], where it was first applied unstably.

The boundary from $C(26,34)$ into the large group $E_2^{2,2j}(22)$ has leading term $\alpha_{m/2}\otimes h^3\iota_{22}$, which is 0 if $\nu<8$. Here we have $m=\frac{1}{2}(j-17)$, and we use the argument of 5.11 to see the factor on the RHS of the $\otimes$. (Because of α_2 and α_1 attaching maps, going from 34 to 22 is like going from 26 to 14, with coefficient $T_4\equiv h^3$ in Proposition 5.11.) The claimed splitting when $\nu<8$ follows for algebraic reasons from Diagram 5.35.

If $\nu=8$, then $\alpha_{m/2}\otimes h^3\iota_{22}$ has order 3 in $E_2^{2,2j}(22)$, by 2.18(1). If we let g denote a generator of $E_2^{2,2j}(34)$, then similarly to Diagram 5.35, $3^3g=a+b$, where a is detected on the 22-class, and b on the 18-class. We have relations in $E_2^{2,2j}(Y_7)$ 3^9a, 3^4b, and (from the boundary) 3^2b+3^8ua, with u a unit in $\mathbf{Z}_{(3)}$. The quotient group is easily seen to be $\mathbf{Z}/3^{12}\oplus\mathbf{Z}/3^2$, with generators g and $(1-3^6u)a-3^3g$. The case $\nu=9$ is extremely similar.

If $\nu\geq 10$, then $E_2^{2,2j}(22)\approx\mathbf{Z}/3^{11}$, and the component of the boundary into this part hits 3^8 times the generator, as before. But this implies now that the class on $E_2^{1,2j}(26)$ hits the element of order 3 in $E_2^{2,2j}(22)$. Whereas in the cases $\nu=8$ and 9, the hitting into the 22-part was without much consequence, because it just adjoined another summand to the classes on the 10-cell which were being hit, the boundary described in the preceding sentence causes one less element in the kernel and cokernel. In the sort of description given in the previous paragraph, the relation 3^9a is changed to $3^{11}a$. Now we have

$$3^{13}g=3^{10}a=-\tfrac{1}{u}3^4b=0,$$

and the claimed splitting follows. □

6. The final case

In this section, we establish the final and most difficult case of $E_2^{1,2j}(Y_7)$, with $\nu(j-17)=13$. We will explain why we cannot say for exactly which such values of j the maximal order is achieved.

Theorem 6.1. *If j is odd, and $\nu(j-17)=13$, then for δ equal to one of the numbers 2, 5, or 8,*

$$E_2^{1,2j}(Y_7)\approx E_2^{2,2j}(Y_7)\approx\mathbf{Z}/3^2\oplus\mathbf{Z}/3^{\min(19,\nu(j-17-2\delta\cdot 3^{13})+4)}.$$

The methods of this paper do not allow us to determine which of the three numbers equals δ.

Proof. Let $j-17=2\cdot 3^{13}c$, with $c\not\equiv 0$ mod 3. The proof begins just like that of the case $j\equiv 8$ mod 9 in Theorem 5.32. In the diagram of the type that we have been using, the main tower has groups of exponent 14, 1, 1, 1, and 2, reading from bottom to top, and a group of exponent 2 extending above the lowest 1. We choose as the generator of $E_2^{1,2j}(34)$ the element $-\alpha_{m/14}\iota_{34}$, where $m=\frac{1}{2}(j-17)$. We use (2.16) to write it as $(h^{14}+L)\iota_{34}$, with L defined on S^{27}. (We choose the minus on α to remove the minus signs in (2.16) and (2.17).)

The boundary $E_2^{1,2j}(34)\xrightarrow{\partial}E_2^{2,2j}(26)$ sends this generator to a class congruent mod lower terms to

$$h^{14}\otimes 2vh\iota_{26}\equiv 2h^{12}\otimes h\iota_{26}\equiv -2h\otimes h^{10}\iota_{26}=d((h^{11}+L')\iota_{26}).$$

Here we use all three parts of 2.21, with the last step using that $j-13=2c3^{13}+4$, and so $\frac{1}{2}(j-13)\equiv 2$ mod a high power of 3. Thus the generator pulls back to $z\equiv h^{14}\iota_{34}-h^{11}\iota_{26}$.

Next we consider $\partial(z)$ in both $E_2^{2,2j}(22)$ and in $E_2^{2,2j}(18)$. The former has leading term $(h^{14}\otimes h^3-h^{11}\otimes(-h))\iota_{22}$. This desuspends to S^{19} and hence is 0 in $E_2^{2,2j}(22)\approx \mathbf{Z}/3^2$. Since the 22-cell factor is split from $C(10,14)$, we do not need to write this as a boundary and append to z. In $C^2(18)$, we have

$$\partial(z)\equiv(h^{14}\otimes\tfrac{1}{2}v^2h^2-h^{11}\otimes 2vh)\iota_{18}\equiv -2h^9\otimes h\iota_{18}\equiv 2h\otimes h^7\iota_{18}=d(-\tfrac{1}{2}h^8\iota_{18}),$$

similarly to the previous paragraph. Thus z pulls back to $z'\equiv z+\frac{1}{2}h^8\iota_{18}$.

The leading term of $\partial(z')$ in $E_2^{2,2j}(14)$ is $\frac{1}{2}h^8\otimes(-h)\iota_{14}\equiv\frac{1}{2}h\otimes h^6\iota_{14}$, and so $\partial(z')=d((-\frac{1}{10}h^7+L)\iota_{14})$, since $\frac{1}{2}(j-7)\equiv 5$ mod a high power of 3. As we will be working at most mod 9, we replace the 10 by 1. Thus z' pulls back to

$$z''\equiv-\alpha_{m/14}\iota_{34}-h^{11}\iota_{26}+\tfrac{1}{2}h^8\iota_{18}+h^7\iota_{14}. \tag{6.2}$$

Now we use (2.17) for $\alpha_{m/14}$, and obtain terms in $\partial(z'')$ due to the first and last terms of (6.2):

$$\partial(z'')\equiv(ch\otimes\tfrac{1}{4}vh^5+h^7\otimes(-h))\iota_{10}\equiv(\tfrac{1}{4}c+1)h\otimes h^5\iota_{10}. \tag{6.3}$$

This is a generator if $c\equiv 1$ mod 3, in which case the diagram described at the beginning of the proof has differential from the generator of $E_2^{2,2j}(34)$ and 3 times the generator killing $E_2^{2,2j}(10)$, yielding $\mathbf{Z}/3^2\oplus\mathbf{Z}/3^{17}$ as the groups $E_2^{1,2j}(Y_7)$ and $E_2^{2,2j}(Y_7)$, as claimed in this case. The splitting is true for algebraic reasons.

If $c\equiv 2$ mod 3, then $\partial(z'')$ is not a generator of $E_2^{2,2j}(10)\approx\mathbf{Z}/9$, but it might be 3 times the generator. This requires second-order information throughout the entire analysis above. This is something that we have not had to do in past applications. In particular, we need finer information in all three parts of Lemma 2.21, in both descriptions of $\alpha_{m/e}$ in Theorem 2.15(2), and in Proposition 5.11.

We now write $c=3k+2$. The cycle z'' above can be written as

$$\begin{aligned} z''=&-\alpha_{m/14}\iota_{34}+(-h^{11}+A_{10}+L_{10})\iota_{26}+(\tfrac{1}{2}h^8+A_7+L_7)\iota_{18}\\ &+(h^7+A_6+L_6)\iota_{14}, \end{aligned} \tag{6.4}$$

where A_i has excess exactly i, and L_i has excess less than i. When we evaluate $\partial(z'')$, the terms of excess 5 will cancel out as in (6.3) with $c\equiv 2$, and so we can desuspend $\partial(z'')$ to S^9. Our differential into $E_2^{2,2j}(10)$ is equal to 3 times the generator if and only if the desuspension of $\partial(z'')$ yields a generator of $E_2^{2,2j-1}(S^9)$.

Let B_4 be the terms of excess exactly 4 in T_3 of Proposition 5.11. The terms of excess 4 or 5 in $\partial(z'')$ are

$$-\alpha_{m/14}\otimes(\tfrac{1}{4}vh^5+B_4)+\tfrac{1}{2}h^8\otimes\tfrac{1}{2}h^2+(h^7+A_6)\otimes(-h). \tag{6.5}$$

Note how certain terms such as ∂A_{10} and ∂A_7 were dropped because they yield terms whose excess is less than 4.

By an analysis similar to [13, 2.11(5)] we have, when $p = 3$ and $c \not\equiv 0$ mod 3,

$$\alpha_{c3^{e-1}/e} \equiv -chv^{c3^{e-1}-1} + \tfrac{3}{2}ch^2v^{c3^{e-1}-2} - 3ch^3v^{c3^{e-1}-3} \bmod 9.$$

Let $h^7 \otimes (-h) = h \otimes h^5 + C_4 + L_4$, where C_4 has excess 4, and L_4 excess less than 4. Omitting terms of excess less than 4, (6.5) becomes

$$\begin{aligned}(6.6)\quad ((3k+2)(h+3h^2) - 3h^2 + 6h^3) \otimes (\tfrac{1}{4}vh^5 + B_4)\\ + \tfrac{1}{4}h^8 \otimes h^2 + h \otimes h^5 + C_4 - A_6 \otimes h,\end{aligned}$$

where the $(h+3h^2)$ comes from $hv^{c3^{e-1}-1} = (v-3h)^{c3^{e-1}-1}h$. Now write $h \otimes vh^5$ as $h \otimes h^5 - 3h^2 \otimes h^5$. Using coefficients of 3 to reduce the excess of terms on the right side of the $\otimes$, we can rewrite (6.6) in excess 4 as

$$(6.7)\quad \tfrac{3}{4}kh \otimes h^5 + \tfrac{3}{2}(h + \tfrac{1}{2}h^2 + h^3) \otimes h^5 + 2h \otimes B_4 + \tfrac{1}{4}h^8 \otimes h^2 + C_4 - A_6 \otimes h,$$

where the $\frac{3}{2}h \otimes h^5$ comes from the $2h \otimes \frac{1}{4}h^5$ and $h \otimes h^5$ in (6.6). Let $D_4 = \frac{3}{2}(h + \frac{1}{2}h^2 + h^3) \otimes h^5 + \frac{1}{4}h^8 \otimes h^2 + C_4$, a specific class of excess 4, independent of the value of k and of any choices of the sort that we are about to mention.

The term B_4 is the terms of excess 4 in (5.17). It could also have included any terms of excess 4 in the homogeneous part of T_3 discussed in the paragraph after (5.17), but as discussed there, this homogeneous part has excess less than 4. Then B_4 contains a term $-\frac{1}{2}v^2h^4$ which appears in (5.17), and it could contain a term $9c_2h^6$ if $c_2 \not\equiv 0$ mod 3. However, because of a term with coefficient $\frac{59}{6}$ which has c_2 as coefficient, we can infer that $c_2 \equiv 0$ mod 3. Thus $B_4 = -\frac{1}{2}v^2h^4$, and so we can let $D_4' = D_4 - h \otimes v^2h^4$, still a specific element of excess 4, and we have

$$(6.8)\qquad \tfrac{1}{4}kh \otimes h^4 + D_4' - A_6 \otimes h$$

as our new expression for $\partial(z'')$ mod L.

Next we study A_6. To find it, we apply $\xrightarrow{\partial} E_2^{2,2j}(14)$ to the sum $z_3 = X_1 + X_2 + X_3$ of the first three terms of (6.4), and write the result as $d(A_6)$. The terms in $\partial(X_1)$ will have excess less than 5, and so may be omitted from the analysis. There is one term, $\frac{1}{2}h^8 \otimes (-h)$, of excess 6, which accounts for the h^7 in (6.2). There are a number of terms of excess 5, which contribute toward A_6. In particular, note that $d(h^6) \equiv h \otimes h^5$, and so each occurrence of $h \otimes h^5$ in $\partial(z_3)$ affects the coefficient of h^6 in A_6. The leading part of $\partial(X_2)$ is $-h^{11} \otimes T_4$, where T_4 is as in 5.11. The full form of T_4 is given in (5.14) and involves a homogeneous part whose coefficient c we do not know. Two parts of this homogeneous part have a factor of 3, which can be used to reduce the excess, but $ch^{11} \otimes v^2h \equiv ch^7 \otimes h \equiv -ch \otimes h^5$ will cause a ch^6-term in A_6, and hence a $ch \otimes h^4$ in (6.8). Thus the coefficient of $h \otimes h^4$ in (6.8) is $k + D + c \in \mathbf{Z}/3$, where D is something which we could compute if we really needed to. Note also that for our purposes (6.8) lies in $\mathbf{Z}/3$ generated by $h \otimes h^4$. The coefficient c has a value; we just don't know how to find it. Therefore, there is one value of k in $\mathbf{Z}/3$ for which (6.8) is 0. (The diligent reader can check that such considerations cannot affect earlier parts of the argument.) Thus the differential into $E_2^{2,2j}(10)$ is 0 if and only if k, defined by $j - 17 = 2(3k+2)3^{13}$, has this value mod 3. Letting $\delta = 3k + 2$ mod 9, this establishes the theorem. $\square$

7. Periodic homotopy of E_7

In this section we use the results for $E_2^{s,2j}(Y_7)$ already achieved to deduce that $v_*(E_7)$ is as claimed in Theorem 1.1. The first result almost finalizes $v_*(Y_7)$, given the results for $E_2^{s,2j}(Y_7)$ determined in the previous two sections.

Theorem 7.1. *The v_1-periodic UNSS of Y_7 converges to $v_*(Y_7)$. If j is odd, then $v_{2j+1}(Y_7) = 0$, $v_{2j}(Y_7) \approx v_{2j}(S^7)$, $v_{2j-2}(Y_7) \approx E_2^{2,2j}(Y_7)$, and there is an exact sequence*

$$0 \to v_{2j-1}(S^7) \to v_{2j-1}(Y_7) \to E_2^{1,2j}(Y_7) \to 0.$$

Proof. One thing that we have to worry about in proving convergence of the v_1-periodic UNSS is to rule out the possibility of a v_1-periodic homotopy class which is not seen in v_1-periodic E_2. This could come about by having a sequence of homotopy classes related by a filtration-increasing v_1-multiplication. We must also rule out the existence of elements in v_1-periodic E_∞ which do not correspond to elements of v_1-periodic homotopy. This could come about from a sequence of E_2-classes related by filtration-preserving v_1-periodicity in E_2, which support arbitrarily large differentials into a sequence of classes related by filtration-increasing v_1-multiplications. The way that we will show that these things cannot happen for Y_7 is to note that Y_7 is built by fibrations from spaces where we have already established convergence.

In (5.7), it was noted how the v_1-periodic UNSS of Y_7 splits into the part from S^7 and the part from even-dimensional classes. As all of this is confined to filtrations 1 and 2, we obtain the following schematic picture for $E_2^{s,t}(Y_7)$, which must necessarily equal E_∞.

$s=2$	ev	S^7			
$s=1$		ev	S^7		
$t-s=$	$2j-2$	$2j-1$	$2j$	$2j+1$	j odd

Here a box labeled S^7 means the corresponding group $E_2^{s,t}(S^7)$, while a box labeled "ev" (for "even") means the corresponding group $E_2^{s,t}(10, 14, 18, 22, 26, 34)$, as computed in Section 5. This E_2 calculation is consistent with the fibrations (5.5) and $S^7 \to \Omega W \to \Omega S^{23}$ of Proposition 5.1.

For $X = \Omega S^{23}$, $\Omega B(11, 15)$, or $\Omega E_7/F_4$, the v_1-periodic UNSS collapses to isomorphisms, if j is odd,

$$v_1^{-1}\pi_{2j+\epsilon}(X) \approx \begin{cases} 0 & \text{if } \epsilon = 0 \text{ or } 1 \\ E_2^{2,2j}(X) & \text{if } \epsilon = -2 \\ E_2^{1,2j}(X) & \text{if } \epsilon = -1. \end{cases}$$

This is true for ΩS^{23} by [9, 6.1], for $\Omega B(11, 15)$ by the fibration

$$\Omega S^{11} \to \Omega B(11, 15) \to \Omega S^{15},$$

and for $\Omega E_7/F_4$ by Theorem 4.6. (Although 4.6 dealt with convergence for E_7/F_4, the methods of Section 5 show that the calculation for $E_2(\Omega E_7/F_4)$ is just that for $E_2(E_7/F_4)$ shifted back by 1 dimension, and of course the same is true of v_1-periodic homotopy groups.)

Let j be odd. We can use a Five Lemma argument once we establish that, for $\epsilon = 1$ or 2, there are morphisms $v_{2j-\epsilon}(-) \to E_2^{\epsilon,2j}(-)$ for these spaces. To see that such morphisms exist, we note that since compact Lie groups and spheres have

H-space exponents ([28]), the spaces with which we deal here all have H-space exponents. By [21], this implies that each v_1-periodic homotopy group is a direct summand of some actual homotopy group, and then we can take the morphism from homotopy to homotopy mod filtration greater than ϵ, which is (unlocalized) E_∞^ϵ, then to (unlocalized) E_2^ϵ as the kernel of the differentials, and then to v_1-periodic E_2^ϵ. This argument is similar to that used in [19].

Thus, letting $X = E_7/F_4$ and $B = B(11,15)$, there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} v_{2j-1}(\Omega X) & \to & v_{2j-2}(\Omega S^{23}\times\Omega B) & \to & v_{2j-2}(Y_7) & \to & v_{2j-2}(\Omega X) & \to & 0 \\ \approx\downarrow & & \approx\downarrow & & \downarrow & & \approx\downarrow & & \\ E_2^{1,2j}(\Omega X) & \to & E_2^{2,2j}(\Omega S^{23}\times\Omega B) & \to & E_2^{2,2j}(Y_7) & \to & E_2^{2,2j}(\Omega X) & \to & 0 \end{array}$$

which implies that $v_{2j-2}(Y_7)\to E_2^{2,2j}(Y_7)$ is an isomorphism.

Similarly, there is a commutative diagram with exact rows and the first column exact

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & E_2^{2,2j+1}(S^7) & \approx & E_2^{2,2j+1}(Y_7) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & v_{2j-1}(\Omega W\times\Omega B) & \to & v_{2j-1}(Y_7) & \to & v_{2j-1}(\Omega X) & \to & v_{2j-2}(\Omega W\times\Omega B) \\ & & \downarrow & & \downarrow & & \approx\downarrow & & \approx\downarrow \\ 0 & \to & E_2^{1,2j}(\Omega S^{23}\times\Omega B) & \to & E_2^{1,2j}(Y_7) & \to & E_2^{1,2j}(\Omega X) & \to & E_2^{2,2j}(\Omega S^{23}\times\Omega B) \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

which implies that the second column fits into a short exact sequence.

The portion of the theorem about $v_{2j+1}(Y_7)$ and $v_{2j}(Y_7)$ is immediate from the exact sequence in $v_*(-)$ associated to the fibration (5.5). $\square$

We restate the following result from [13, 1.3(1)].

Lemma 7.2. *The projection map $B(3,7)\to S^7$ induces an isomorphism in $v_{2j-1}(-)$ unless j is odd and $j\equiv 21$ mod 27, in which case it is a surjection $\mathbf{Z}/3^4\to\mathbf{Z}/3^3$. The isomorphic groups are 0 if j is even, while if j is odd, they are cyclic of order $3^{\min(3,1+\nu(j-3))}$.*

The next result, combined with the above results and Theorems 5.18, 5.23, 5.29, 5.32, and 6.1 gives $v_*(E_7)$ for most values of $*$.

Theorem 7.3. (a) *If j is odd, $j\not\equiv 2$ mod 9, and $j\not\equiv 21$ mod 27, then the exact sequence of the fibration $Y_7\to B(3,7)\to E_7$ breaks up into isomorphisms*

$$v_{2j}(Y_7)\xrightarrow{\approx}v_{2j}(B(3,7)) \quad \text{and} \quad v_{2j-1}(E_7)\xrightarrow{\approx}v_{2j-2}(Y_7)$$

and a short exact sequence

$$0\to v_{2j}(E_7)\to v_{2j-1}(Y_7)\xrightarrow{\phi}v_{2j-1}(B(3,7))\to 0.$$

If $E_2^{1,2j}(Y_7)\approx\mathbf{Z}/3^{e_1}\oplus\mathbf{Z}/3^m$, with $1\le e_1\le 2$, is as given in Theorems 5.18, 5.23, 5.29, 5.32, and 6.1, and $v_{2j-1}S^7\approx v_{2j-1}(B(3,7))\approx\mathbf{Z}/3^{e_2}$ is as in 7.2, then

$$v_{2j-1}(Y_7)\approx\mathbf{Z}/3^{e_1+e_2}\oplus\mathbf{Z}/3^m, \tag{7.4}$$

and ϕ sends the first summand onto $\mathbf{Z}/3^{e_2}$.
(b) *If j is even, then $v_{2j}E_7 = v_{2j-1}E_7 = 0$.*

Note that even if ϕ sent the second summand nontrivially, its kernel would still be $\mathbf{Z}/3^{e_1} \oplus \mathbf{Z}/3^m$, since $m \geq e_1 + e_2$. Thus if j is as in Theorem 7.3(a.), there are abstract isomorphisms $v_{2j}(E_7) \approx E_2^{1,2j}(Y_7)$ and $v_{2j-1}(E_7) \approx E_2^{2,2j}(Y_7)$, with $E_2^{s,2j}(Y_7)$ as given in Theorems 5.18, 5.23, 5.29, 5.32, and 6.1. This implies Theorem 1.1 in these cases.

Proof of Theorem 7.3. There is a commutative diagram of fibrations

$$\begin{array}{ccccc} \Omega W & \to & B(3,7) & \to & K \\ \downarrow & & \Big\downarrow{=} & & \downarrow \\ Y_7 & \to & B(3,7) & \to & E_7 \end{array} \tag{7.5}$$

where the last map is the composite $K \to F_4 \to E_7$. Since by [10, 2.10(i)] the composite $S^7 \to \Omega W \to B(3,7) \to S^7$ has degree 3, we deduce the same of the composite $S^7 \to Y_7 \to B(3,7) \to S^7$. We already know that $v_{2j}S^7 \to v_{2j}Y_7$ is an isomorphism, and $v_{2j}B(3,7) \to v_{2j}S^7$ is multiplication by 3 on isomorphic groups. It follows that $v_{2j}Y_7 \to v_{2j}B(3,7)$ is an isomorphism.

There is a commutative diagram of fibrations

$$\begin{array}{ccccc} S^7 & \to & \Omega W & \to & \Omega S^{23} \\ \downarrow & & \downarrow & & \\ S^7 & \to & Y_7 & & \end{array} \quad . \tag{7.6}$$

The cyclic extension in $v_{2j-1}(\Omega W)$ was established in [10, pp.294-5]. This implies the nontrivial extension in $v_{2j-1}Y_7$ claimed in the theorem from the $\mathbf{Z}/3^{e_1}$ on the 22-class in $E_2^{1,2j}(Y_7)$ to $v_{2j-1}(S^7)$ in the exact sequence of Theorem 7.1.

There cannot be an extension in $v_{2j-1}Y_7$ from the $\mathbf{Z}/3^m$-summand of $E_2^{1,2j}Y_7$ because of the splitting $F_4 = K \times B(11,15)$. The element of order 3 in the large summand of $v_{2j-1}Y_7$ comes from $B(11,15)$, while the S^7 lies in K. This is made explicit in the commutative diagram of fibrations

$$\begin{array}{ccccc} \Omega B(11,15) \times \Omega W & \to & B(3,7) & \to & F_4 \\ \downarrow & & \downarrow & & \downarrow \\ Y_7 & \to & B(3,7) & \to & E_7 \end{array} \tag{7.7}$$

That ϕ sends the first summand of (7.4) onto $v_{2j-1}B(3,7)$ follows from the diagram (7.7) and the surjectivity of $v_{2j-1}(\Omega W) \to v_{2j-1}B(3,7)$ established in [10, pp.297-8]. □

One of the cases omitted in the previous theorem is covered in the following result, the proof of which is very similar.

Theorem 7.8. *If j is odd and $j \equiv 21$ mod 27, then the exact sequence (with $B = B(3,7)$)*

$$0 \to v_{2j}Y_7 \xrightarrow{\phi_1} v_{2j}B \to v_{2j}E_7 \to v_{2j-1}Y_7 \xrightarrow{\phi_2} v_{2j-1}B \to v_{2j-1}E_7 \to v_{2j-2}Y_7 \to 0$$

has ϕ_1 an injection $\mathbf{Z}/3^3 \hookrightarrow \mathbf{Z}/3^4$, and ϕ_2 a surjection from the first summand in $\mathbf{Z}/3^4 \oplus \mathbf{Z}/3^m \to \mathbf{Z}/3^4$. Moreover,

$$v_{2j}E_7 \approx \operatorname{coker}\phi_1 \oplus \ker\phi_2 \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^m.$$

Proof. Similarly to the previous proof, the morphism ϕ_1 follows from [13, 2.5] and [10, 2.10(i)], the structure of $v_{2j-1}Y_7$ follows from (7.6), and the morphism ϕ_2 follows from (7.7). The $\mathbf{Z}/3^m$ in $v_{2j-1}Y_7$ cannot extend cyclically with $\operatorname{coker}\phi_1$ in $v_{2j}E_7$ because the element of order 3 in $\mathbf{Z}/3^m$ lies in $v_{2j-1}\Omega B(11,15)$, while $\operatorname{coker}\phi_1$ lies in $v_{2j}K$, and these cannot be related by a $\cdot 3$-extension due to the splitting $F_4 = K \times B(11,15)$. $\square$

We begin working toward determination of $v_{2j-\epsilon}E_7$ when $j \equiv 2 \bmod 9$ with the following proposition.

Proposition 7.9. *If j is odd and $j \equiv 2$ mod 9, then the exact sequence of the fibration $Y_7 \to B(3,7) \to E_7$ yields*

- $v_{2j}Y_7 \to v_{2j}B(3,7)$ *is an isomorphism of* $\mathbf{Z}/3$*'s;*
- $v_{2j-1}E_7 \to v_{2j-2}Y_7$ *is an isomorphism;*
- $v_{2j}E_7 \approx \ker(v_{2j-1}Y_7 \xrightarrow{\phi} v_{2j-1}B(3,7) \approx \mathbf{Z}/3)$.

Proof. Surjectivity of $v_{2j-1}Y_7 \to v_{2j-1}B(3,7)$ follows from (7.7), while $v_{2j}Y_7 \to v_{2j}B(3,7)$ is bijective as in the proof of 7.3. $\square$

By 7.9, 7.1, and 5.34, $v_{2j-1}E_7$ is seen to be as claimed in Theorem 1.1 when $j \equiv 2$. It remains to determine $v_{2j-1}Y_7$ and ϕ, from which $v_{2j}E_7$ follows.

Theorem 7.10. *Let j be odd, and $\nu = \nu(j-11)$. If $2 \le \nu \le 9$, then*

$$v_{2j-1}Y_7 \approx \mathbf{Z}/3^{\nu+3} \oplus \mathbf{Z}/3^4$$

and ϕ sends $\mathbf{Z}/3^4$ nontrivially. Thus $\ker\phi \approx \mathbf{Z}/3^{\nu+3} \oplus \mathbf{Z}/3^3$*, regardless of* $\phi \mid \mathbf{Z}/3^{\nu+3}$.

Proof. Similarly to the proof of Theorem 7.3, the extension in

$$\mathbf{Z}/3 \approx v_{2j-1}S^7 \to v_{2j-1}Y_7 \to E_2^{1,2j}Y_7 \approx \mathbf{Z}/3^{\nu+2} \oplus \mathbf{Z}/3^4$$

is nontrivial from the first summand. From [10, 2.12], $v_{2j-1}(\Omega W) \to v_{2j-1}B(3,7)$ is a surjection $\mathbf{Z}/3^{\nu+2} \to \mathbf{Z}/3$, and from (7.5) it factors as

$$v_{2j-1}(\Omega W) \to v_{2j-1}Y_7 \xrightarrow{\phi} v_{2j-1}B(3,7).$$

From (7.6), $v_{2j-1}(\Omega W) \to v_{2j-1}Y_7$ is an injection $\mathbf{Z}/3^{\nu+2} \to \mathbf{Z}/3^{\nu+3} \oplus \mathbf{Z}/3^4$, since the element of order 3 in $v_{2j-1}(\Omega W)$, which comes from $v_{2j-1}(S^7)$, maps nontrivially. The result now follows from elementary algebra. $\square$

The same ingredients imply the following result.

Theorem 7.11. *If j is odd and $\nu(j-11) \ge 10$, but $j \not\equiv 11 + 2 \cdot 3^{10}$ mod $2 \cdot 3^{11}$, then*

$$v_{2j-1}Y_7 \approx \mathbf{Z}/3^{12} \oplus \mathbf{Z}/3^4$$

and ϕ is surjective in Proposition 7.9.

We cannot deduce from this which summand(s) of $v_{2j-1}Y_7$ maps nontrivially under ϕ, and so we cannot tell whether $\ker\phi$ is $\mathbf{Z}/3^{12} \oplus \mathbf{Z}/3^3$ or $\mathbf{Z}/3^{11} \oplus \mathbf{Z}/3^4$. We suspect that $\mathbf{Z}/3^4$ maps across, which would imply the first splitting.

Finally we have the following result in the exceptional case. In order to keep the statement of Theorem 1.1 readable, we did not distinguish there between this case, in which we know the precise structure of $v_{2j-1}E_7$, and the case of Theorem 7.11, where we do not.

Theorem 7.12. *If* $j \equiv 11 + 2 \cdot 3^{10}$ *mod* $2 \cdot 3^{11}$, *then*

$$v_{2j}E_7 \approx \mathbf{Z}/3^{12} \oplus \mathbf{Z}/3^3.$$

Proof. As in the proof of 7.10, $v_{2j-1}Y_7 \approx \mathbf{Z}/3^{12} \oplus \mathbf{Z}/3^4$. It was shown in [10, 2.12] that $v_{2j-1}(\Omega W) \to v_{2j-1}B(3,7)$ is 0 if $j \equiv 11 + 2 \cdot 3^{10}$ mod $2 \cdot 3^{11}$.

Let G denote the fiber of $K \to E_7$. There is a commutative diagram of fibrations

$$\begin{array}{ccccc}
\Omega W & \xrightarrow{=} & \Omega W & & \\
\downarrow & & \downarrow & & \\
Y_7 & \to & B(3,7) & \to & E_7 \\
\downarrow & & \downarrow & & \downarrow \\
G & \to & K & \to & E_7.
\end{array}$$

It follows from the Serre spectral sequence of the fibration $\Omega W \to Y_7 \to G$ that

$$BP_*(G) \approx BP_*[x_{10}, x_{14}, x_{18}, x_{26}, x_{34}],$$

and so charts for v_*G are like charts for $E_2^{s,2j}(Y_7)$ without the part on the 22-class. The chart for $v_{2j-1}G$ and $v_{2j-2}G$ whenever $j \equiv 2$ mod 9 is like Diagram 5.35 without the $\nu + 1$. In particular, $v_{2j-1}G$ is cyclic with generator on the 26-class. The proof of Theorem 5.34 in the case $\nu \geq 10$, where it says that the class on $E_2^{1,2j}(26)$ hits the element of order 3 in $E_2^{2,2j}(22)$, implies that $v_{2j-1}G \to v_{2j-1}K$ sends the generator to the element of order 3^2. Now it follows from the following commutative diagram with exact rows that ϕ is surjective on the $\mathbf{Z}/3^4$ summand.

$$\begin{array}{ccccccc}
 & \mathbf{Z}/3^{11} & & \mathbf{Z}/3^{12} \oplus \mathbf{Z}/3^4 & & \mathbf{Z}/3^5 & & \\
0 \to & v_{2j}W & \to & v_{2j-1}Y_7 & \to & v_{2j-1}G & \to & v_{2j-1}W \\
 & & & \Big\downarrow \phi & & \downarrow & & \Big\downarrow = \\
 & & & v_{2j-1}B & \to & v_{2j-1}K & \to & v_{2j-1}W \\
 & & & \mathbf{Z}/3 & & \mathbf{Z}/3^{12} & & \mathbf{Z}/3^{11}
\end{array}$$

□

References

[1] M. Arkowitz, *Localization and H-spaces*, Aarhus Univ. Lecture Note Series, No. 44, Matematisk Institut, Aarhus Universitet, Aarhus, 1976, MR 53 #4048.

[2] M.F. Atiyah, *Thom complexes*, Proc. London Math. Soc. **11** (1961) 291–310, MR 24 #A1727.

[3] M. Bendersky, *Unstable towers in the odd primary homotopy groups of spheres*, Trans. Amer. Math. Soc. **276** (1985) 529–542, MR 86g:55018.

[4] M. Bendersky, *The v_1-periodic unstable Novikov spectral sequence*, Topology **31** (1992) 47–64, MR 93i:55026.

[5] M. Bendersky, *Some calculations in the unstable Adams-Novikov spectral sequence*, Publ. Res. Inst. Math. Sci. **16** (1980) 739–766, MR 82f:55028.

[6] M. Bendersky, *The derived functors of the primitives for $BP_*(\Omega S^{2n+1})$*, Trans. Amer. Math. Soc. **276** (1983) 599–619, MR 85f:55008a.

[7] M. Bendersky, *The BP Hopf invariant*, Amer. Jour. Math. **108** (1986) 1037–1058, MR 88d:55015.

[8] M. Bendersky, E. B. Curtis, and H.R. Miller, *The unstable Adams-Novikov spectral sequence for generalized homology*, Topology **17** (1978) 229–248, MR 80g:55035.

[9] M. Bendersky, E. B. Curtis, and D.C. Ravenel, *The EHP sequence in BP-theory*, Topology **21** (1982) 373–391, MR 83k:55009.

[10] M. Bendersky and D. M. Davis, *3-primary v_1-periodic homotopy groups of F_4 and E_6*, Trans. Amer. Math. Soc. **344** (1994) 291–306, MR 95d:55013.

[11] M. Bendersky and D. M. Davis, *2-primary v_1-periodic homotopy groups of $SU(n)$*, Amer. J. Math. **114** (1991) 465–494, MR 93g:55018.

[12] M. Bendersky, D. M. Davis, and M. Mahowald, *v_1-periodic homotopy groups of $Sp(n)$*, Pacific Journal of Math. **170** (1995) 319–378, MR 96h:55011.

[13] M. Bendersky, D. M. Davis, and M. Mimura, *v_1-periodic homotopy groups of exceptional Lie groups: torsion-free cases*, Trans. Amer. Math. Soc. **333** (1992) 115–135, MR 92k:57071.

[14] M. Bendersky and R. D. Thompson, *The Bousfield-Kan spectral sequence for periodic homology theories*, to appear.

[15] M. Bendersky and R. D. Thompson, *Some properties of the K-theory completion*, to appear.

[16] A. K. Bousfield, *Nice homology coalgebras*, Trans. Amer. Math. Soc. **148** (1970) 473–489, MR 41 #3564.

[17] A. K. Bousfield, *The K-theory localizations and v_1-periodic homotopy groups of H-spaces*, to appear.

[18] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *The double suspension and exponents of the homotopy groups of spheres*, Annals of Math. **110** (1979) 549–565, MR 81c:55021.

[19] D. M. Davis, *v_1-periodic homotopy groups of $SU(n)$ at odd primes*, Proc. London Math. Society **43** (1991) 529–544, MR 92d:55016.

[20] D. M. Davis, *Elements of large order in $\pi_*(SU(n))$*, Topology **37** (1998) 293–327.

[21] D. M. Davis and M. Mahowald, *Some remarks on v_1-periodic homotopy groups*, Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990), London Math. Soc. Lecture Notes Series, No. 176, Cambridge Univ. Press, Cambridge, 1992, pp. 55–72, MR 94h:55015.

[22] D. M. Davis and M. Mahowald, *Three contributions to the homotopy theory of the exceptional Lie groups G_2 and F_4*, Jour. Math. Soc. Japan **43** (1991) 55–72, MR 92g:57048.

[23] D. M. Davis and H. Yang, *Tractable formulas for v_1-periodic homotopy groups of $SU(n)$ when $n \leq p^2 - p + 1$*, Forum Math **8** (1996) 585–619, MR 97m:55018.

[24] V. Giambalvo, *Some tables for formal groups and BP*, Lecture Notes in Math., No. 658, Springer-Verlag, Berlin, 1978, pp. 169–176, MR 80d:55004.

[25] B. Gray, *On the sphere of origin of infinite families in the homotopy groups of spheres*, Topology **8** (1969) 219–232, MR 39 #6321.

[26] H. Hamanaka and S. Hara, *The mod 3 homology of the space of loops on the exceptional Lie groups, and the adjoint action*, Jour. Math. Kyoto Univ. **37** (1997) 441–454.

[27] J. Harper, *H-spaces with torsion*, Memoirs Amer. Math. Soc., No. 223, Amer. Math. Soc., Providence, 1979, MR 80k:55033.

[28] I. M. James, *On Lie groups and their homotopy groups*, Proc. Camb. Phil. Soc. **55** (1959) 244–247, MR 23 #A1371.

[29] A. Kono and M. Mimura, *Cohomology operations and the Hopf algebra structures of the compact exceptional Lie groups E_7 and E_8*, Proc London Math Soc **35** (1977) 345-358.

[30] M. Mimura, G. Nishida, and H. Toda, *Mod p decomposition of compact Lie groups*, Publ. Res. Inst. Math. Sci. **13** (1977) 627–680, MR 57 #17675.

[31] D. C. Ravenel and W. S. Wilson, *The Hopf ring for complex cobordism*, J. Pure and Appl. Algebra **9** (1976/77) 241–280, MR 56 #6644.

[32] V.P. Snaith, *On the K-theory of homogeneous spaces and conjugate bundles of Lie groups*, Proc. London Math. Soc. **22** (1971) 562–584, MR 44 #7584.

[33] W. S. Wilson, *The Ω-spectrum for Brown-Peterson homology*, I, Comm. Math. Helv. **48** (1973) 45–55, MR 48 #5055.

[34] N. Yagita, *The BP^*-module structure of $BP^*(E_8)$ for $p = 3$*, Publ. Sec. Mat. Univ. Autonoma Barcelona **28** (1984) 5–18, MR 87h:57037.

Lehigh University, Bethlehem, PA 18015
dmd1@lehigh.edu http://www.lehigh.edu/~dmd1/dmd1.html

This paper is available via http://nyjm.albany.edu:8000/j/1998/4-13.html.