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Heisenberg Lie Bialgebras as Central Extensions

Miloud Benayed and **El Mamoun Souidi**

Abstract. We determine and study all Lie bialgebra central extensions of $\mathbb{R}^{2n}$ by $\mathbb{R}$ admitting the Heisenberg algebra $\mathcal{H}_{2n+1}$ as the underlying Lie algebra structure. The present work answers the question about the realizability of Lie bialgebra structures on $\mathcal{H}_{2n+1}$ as central extensions of $\mathbb{R}^{2n}$ endowed with the adapted Lie bialgebra structure, by $\mathbb{R}$.

Contents

1. Introduction

A Lie bialgebra is a Lie algebra $(\mathfrak{g}, [\,,\,])$ equipped with a 1-cocycle $\varepsilon : \mathfrak{g} \to \wedge^2\mathfrak{g}$ ($\delta\varepsilon = 0$) for the extended adjoint action of $\mathfrak{g}$ on $\wedge^2\mathfrak{g}$, whose transpose $\varepsilon^* : \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie algebra structure on the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$. A 1-coboundary $\varepsilon = \delta r$ with $r \in \wedge^2\mathfrak{g}$ defines a Lie bialgebra structure on a Lie algebra $(\mathfrak{g}, [\,,\,])$ if and only if the Schouten bracket $[\![r, r]\!] \in \wedge^3\mathfrak{g}$ is invariant under the extended adjoint action of $\mathfrak{g}$ on $\wedge^3\mathfrak{g}$. Such a Lie bialgebra is called coboundary or exact.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie bialgebras. A linear map $\rho : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie bialgebra morphism if ρ is a Lie algebra morphism and its transpose $\rho^* : \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ is also a Lie algebra morphism. A bijective morphism is called an isomorphism.

Two Lie bialgebra structures ε_1 and ε_2 on a Lie algebra $(\mathfrak{g}, [\;,\;])$ are called equivalent if there exists a Lie bialgebra isomorphism $\rho : (\mathfrak{g}, [\,,\,], \varepsilon_1) \to (\mathfrak{g}, [\,,\,], \varepsilon_2)$.

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For further details on Lie bialgebras we refer the reader to [3]. In all the sequel the ground field is $\mathbb{R}$.

A Lie bialgebra $\widehat{\mathfrak{g}}$ is called a central extension of a Lie bialgebra $\mathfrak{g}$ by $\mathbb{R}$ if there exists an exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ i\ } \widehat{\mathfrak{g}} \xrightarrow{\ \pi\ } \mathfrak{g} \longrightarrow 0$$

in which i and π are Lie bialgebra morphisms and $i(\mathbb{R})$ is contained in the center of the Lie algebra $\widehat{\mathfrak{g}}$. Two central extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by $\mathbb{R}$ will be called equivalent if there exists a Lie bialgebra morphism $\rho : \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ inducing the identity on $\mathbb{R}$ and $\mathfrak{g}$, in the exact sequences defining $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$.

The Heisenberg algebra $\mathcal{H}_{2n+1}$ is a Lie algebra central extension of the abelian Lie algebra $\mathbb{R}^{2n}$ by $\mathbb{R}$, associated with the 2-cocycle γ which is the canonical symplectic 2-form of $\mathbb{R}^{2n}$. A Lie bialgebra structure on the abelian Lie algebra $\mathbb{R}^{2n}$ is equivalent to the data of a Lie algebra structure on $(\mathbb{R}^{2n})^*$.

This paper is organized as follows. In Section 2, we describe the set $\mathrm{Ext}_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ of all inequivalent Lie bialgebra central extensions of a given Lie bialgebra structure on the abelian Lie algebra $\mathbb{R}^{2n}$, by $\mathbb{R}$. Let $\mathrm{Ext}^{\gamma}_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ be the subset consisting of elements in $\mathrm{Ext}_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ which have the Heisenberg algebra as the underlying Lie algebra structure. Such Lie bialgebra structures on $\mathcal{H}_{2n+1}$ will be called central. We prove that $\mathrm{Ext}^{\gamma}_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ is non empty if and only if $(\mathbb{R}^{2n})^*$ is an abelian Lie algebra. If the last condition is fulfilled then the set $\mathrm{Ext}^{\gamma}_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ is parametrized by the endomorphisms of $\mathbb{R}^{2n}$. Section 3 is devoted to give a characterization of exact central structures on $\mathcal{H}_{2n+1}$ by means of the associated endomorphims of $\mathbb{R}^{2n}$. In Section 4 we give the orbits of central structures on $\mathcal{H}_{2n+1}$ under its automorphisms group action. In the last section we give a motivation of this work as a partial answer of an open question.

2. Central Structures on $\mathcal{H}_{2n+1}$

We endow $\mathbb{R}^{2n}$ with a Lie bialgebra structure by taking the abelian Lie algebra structure on $\mathbb{R}^{2n}$ and giving a Lie algebra structure $[\,,\,]$ on its dual vector space $(\mathbb{R}^{2n})^*$. Let "ad" be the adjoint action of $(\mathbb{R}^{2n})^*$ on itself and "coad" represents the coadjoint action of $(\mathbb{R}^{2n})^*$ on its dual vector space $\mathbb{R}^{2n}$. We denote by $\mathrm{Der}(\mathbb{R}^{2n})^*$ the vector space of derivations of the Lie algebra $(\mathbb{R}^{2n})^*$ and $\mathrm{Dex}(\mathbb{R}^{2n})^*$ stands for the outer derivations of $(\mathbb{R}^{2n})^*$.

Definition 2.1. An element ω of $\wedge^2(\mathbb{R}^{2n})^*$ is called compatible with the Lie algebra structure on $(\mathbb{R}^{2n})^*$ if the condition: $\mathrm{coad}_{\widetilde{\omega}(x)}(y) = \mathrm{coad}_{\widetilde{\omega}(y)}(x)$ holds for all x, y in $\mathbb{R}^{2n}$, where $\widetilde{\omega} : \mathbb{R}^{2n} \to (\mathbb{R}^{2n})^*$ is the linear map induced by ω. We let $\wedge^2_c(\mathbb{R}^{2n})^*$ to denote the subspace of all such compatible elements of $\wedge^2(\mathbb{R}^{2n})^*$.

Theorem 2.2. *There is a one-to-one correspondence between* $\mathrm{Ext}_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ *and* $\wedge^2_c(\mathbb{R}^{2n})^* \times \mathrm{Dex}(\mathbb{R}^{2n})^*$.

Proof. To any Lie bialgebra central extension $\mathbb{R}^{2n} \oplus \mathbb{R}$ of $\mathbb{R}^{2n}$ by $\mathbb{R}$ one can associate a couple $(\omega, f) \in \wedge^2_c(\mathbb{R}^{2n})^* \times \mathrm{Der}(\mathbb{R}^{2n})^*$ as follows (see [1]):

$$[(x, a); (y, b)]_{\mathbb{R}^{2n} \oplus \mathbb{R}} = (0; \omega(x, y))$$

$$[(\xi, \alpha); (\eta, \beta)]_{(\mathbb{R}^{2n})^* \oplus \mathbb{R}} = ([\xi, \eta]_{(\mathbb{R}^{2n})^*} + \alpha f(\eta) - \beta f(\xi); 0)$$

Two elements (ω, f) and (ω', f') of $\wedge_c^2(\mathbb{R}^{2n})^* \times \mathrm{Der}(\mathbb{R}^{2n})^*$ define equivalent Lie bialgebras if and only if $\omega' = \omega$ and $f' = f + \mathrm{ad}_\varphi$ where φ lies in $(\mathbb{R}^{2n})^*$. Reversing the arguments we also get the converse. $\square$

For every $\omega \in \wedge_c^2(\mathbb{R}^{2n})^*$ we let $\mathrm{Ext}^\omega_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ to be the subspace of $\mathrm{Ext}_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ corresponding to $\{\omega\} \times \mathrm{Dex}(\mathbb{R}^{2n})^*$. We are interested in the case where ω is the canonical symplectic 2-form γ of $\mathbb{R}^{2n}$:

$$\gamma : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$$
$$((x, y); (x', y')) \mapsto x \cdot y' - y \cdot x'$$

The dots stand here for the inner product in $\mathbb{R}^{2n}$. In order to restrict ourselves to $\mathrm{Ext}^\gamma_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$, we must verify the compatibility of γ with the given Lie algebra structure on $(\mathbb{R}^{2n})^*$.

Lemma 2.3. *The following conditions are equivalent:*
 (i) $\gamma \in \wedge_c^2(\mathbb{R}^{2n})^*$.
 (ii) $(\mathbb{R}^{2n})^*$ *is an abelian Lie algebra.*

Proof. It is enough to prove the implication (i) $\Rightarrow$ (ii). The fact that γ lies in $\wedge_c^2(\mathbb{R}^{2n})^*$ implies the following condition:

$$\forall x \in \mathbb{R}^{2n},\ \forall \xi \in (\mathbb{R}^{2n})^*, \quad \widetilde{\gamma}(\mathrm{coad}_\xi\, x) = [\xi, \widetilde{\gamma}(x)].$$

Writing this condition for $\xi = \widetilde{\gamma}(y)$ with $y \in \mathbb{R}^{2n}$ and using the fact that $\gamma \in \wedge_c^2(\mathbb{R}^{2n})^*$, we obtain

$$\forall x, y \in \mathbb{R}^{2n}, \quad [\widetilde{\gamma}(x), \widetilde{\gamma}(y)] = [\widetilde{\gamma}(y), \widetilde{\gamma}(x)]$$

which implies that $[\widetilde{\gamma}(x), \widetilde{\gamma}(y)] = 0$ for all x, y in $\mathbb{R}^{2n}$. The non-degeneracy of γ means that $\widetilde{\gamma} : \mathbb{R}^{2n} \to (\mathbb{R}^{2n})^*$ is a vector space isomorphism. So we conclude that $(\mathbb{R}^{2n})^*$ is an abelian Lie algebra. $\square$

Remark 2.4. The condition $\omega \in \wedge_c^2(\mathbb{R}^{2n})^*$ implies that the range of $\widetilde{\omega}$ is an abelian Lie subalgebra of $(\mathbb{R}^{2n})^*$. The converse is false in general, as one can see in the following example. We endow $(\mathbb{R}^4)^*$ with the Lie algebra structure defined by the only non vanishing bracket $[X_2^*, X_3^*] = X_4^*$, where $(X_k^*)_{k=1}^4$ is the dual basis of the canonical ordered basis $(X_k)_{k=1}^4$ of $\mathbb{R}^4$. Let ω be the element of $\wedge^2(\mathbb{R}^4)^*$ given by $\omega(X_1, X_2) = 1$ and vanishing elsewhere. The range of $\widetilde{\omega}$ is abelian but $\omega \notin \wedge_c^2(\mathbb{R}^4)^*$, for example $\mathrm{coad}_{\widetilde{\omega}(X_1)}(X_4) = -X_3 \neq 0 = \mathrm{coad}_{\widetilde{\omega}(X_4)}(X_1)$.

As a consequence of the previous lemma and theorem we get the following.

Proposition 2.5. $\mathrm{Ext}^\gamma_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ *is non empty if and only if* $(\mathbb{R}^{2n})^*$ *is an abelian Lie algebra. If the last condition is fulfilled then* $\mathrm{Ext}^\gamma_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ *is parametrized by the space* $\mathrm{End}(\mathbb{R}^{2n})$ *of all endomorphisms of* $\mathbb{R}^{2n}$*.*

Henceforth we assume that $\mathbb{R}^{2n}$ is an abelian Lie bialgebra, i.e., $\mathbb{R}^{2n}$ and $(\mathbb{R}^{2n})^*$ are endowed with the zero Lie brackets. Otherwise, as we have seen, $\mathrm{Ext}^\gamma_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ will be empty.

The set $\mathrm{Ext}^\gamma_{\mathrm{big}}(\mathbb{R}^{2n}, \mathbb{R})$ is viewed as the Heisenberg algebra $\mathcal{H}_{2n+1}$ endowed with Lie bialgebra structures, parametrized by $\mathrm{End}(\mathbb{R}^{2n})$, such that each structure makes $\mathcal{H}_{2n+1}$ as a central extension of the abelian Lie bialgebra $\mathbb{R}^{2n}$ by $\mathbb{R}$. Such a structure on $\mathcal{H}_{2n+1}$ will be called central. Let us specify the central structures on $\mathcal{H}_{2n+1}$ in terms of the corresponding 1-cocycles $\varepsilon : \mathcal{H}_{2n+1} \to \wedge^2 \mathcal{H}_{2n+1}$ (the transposes of Lie brackets in $\mathcal{H}^*_{2n+1}$: See the proof of Theorem 2.2).

The center $\mathcal{Z}(\mathcal{H}_{2n+1})$ of the Lie algebra $\mathcal{H}_{2n+1}$ is one dimensional, let Z be a non zero element of $\mathcal{Z}(\mathcal{H}_{2n+1})$. Let $(X_k)_{k=1}^{2n}$ be the canonical ordered basis of $\mathbb{R}^{2n}$ then $((X_k)_{k=1}^{2n}, Z)$ is (the canonical) basis of $\mathcal{H}_{2n+1}$. The only non vanishing Lie brackets on $\mathcal{H}_{2n+1}$ are given by:

$$\text{For all } 1 \le i, j \le n \ , \ [X_i, X_{n+j}] = \delta_{ij} Z$$

where δ_{ij} is the Kronecker's symbol.

Definition 2.6. A central structure on $\mathcal{H}_{2n+1}$ is the data of a 1-cocycle ε_f : $\mathcal{H}_{2n+1} \to \wedge^2 \mathcal{H}_{2n+1}$ where $f \in \mathrm{End}(\mathbb{R}^{2n})$ satisfying:

(i) $\varepsilon_f(Z) = 0$.
(ii) For all $X \in \mathbb{R}^{2n}$, $\varepsilon_f(X) = Z \wedge f(X)$.

3. Exact Central Structures on $\mathcal{H}_{2n+1}$

An arbitrary element r of $\wedge^2 \mathcal{H}_{2n+1}$ can be written as

$$r = \sum_{1 \le i < j \le 2n} \alpha_{ij} X_i \wedge X_j + \sum_{i=1}^{2n} \beta_i X_i \wedge Z$$

where $\alpha = (\alpha_{ij})_{1 \le i,j \le 2n}$ is an antisymmetric matrix and the β_i are in $\mathbb{R}$.

Lemma 3.1. *The couboundary $\delta r'$ of $r' = \sum_{i=1}^{2n} \beta_i X_i \wedge Z$ vanishes.*

Proof. For all $H \in \mathcal{H}_{2n+1}$ we have:

$$(\delta r')(H) = \sum_{i=1}^{2n} \beta_i([H, X_i] \wedge Z + X_i \wedge [H, Z]).$$

Since $[\mathcal{H}_{2n+1}, \mathcal{H}_{2n+1}] = \mathcal{Z}(\mathcal{H}_{2n+1})$ and Z is central in the Lie algebra $\mathcal{H}_{2n+1}$ then $(\delta r')(H) = 0$, for all $H \in \mathcal{H}_{2n+1}$. □

Henceforth we assume that $r = \sum_{1 \le i < j \le 2n} \alpha_{ij} X_i \wedge X_j$. The matrix α determines r completely and vice-versa.

Lemma 3.2. [2] *Every element r of $\wedge^2 \mathcal{H}_{2n+1}$ defines an exact Lie bialgebra structure δr on $\mathcal{H}_{2n+1}$, i.e., the Schouten bracket $[\![r, r]\!]$ is* ad*-invariant, for all r in $\wedge^2 \mathcal{H}_{2n+1}$.*

Let us remark that an exact Lie bialgebra δr on $\mathcal{H}_{2n+1}$ is necessarily central. Our aim is to give a characterization of central structures on $\mathcal{H}_{2n+1}$ that are exact.

Proposition 3.3. *An endomorphism f of $\mathbb{R}^{2n}$ defines an exact central structure on $\mathcal{H}_{2n+1}$ if and only if the matrix of f (in the basis $(X_k)_{k=1}^{2n}$) has the form $\begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}$ where A, B, C are $n \times n$ matrices with B and C antisymmetric. The corresponding r-matrix has the form $\alpha = \begin{pmatrix} -B & A \\ -{}^tA & C \end{pmatrix}$.*

Proof. Let $r = \sum_{1 \le i<j \le 2n} \alpha_{ij} X_i \wedge X_j$ be an element of $\wedge^2 \mathcal{H}_{2n+1}$. Set $\alpha = (\alpha_{ij})$. We verify there exists an $f \in \mathrm{End}(\mathbb{R}^{2n})$ with matrix (f_{ij}) (in the canonical basis) such that $\delta r = \varepsilon_f$ if and only if $f_{ik} = \alpha_{i,n+k}$, $f_{n+k,k} = 0$ for all $1 \le k \le n$, and $f_{ik} = \alpha_{k-n,i}$, $f_{k-n,k} = 0$ for all $n+1 \le k \le 2n$. Using these relations and the antisymmetry of the matrix α, we obtain the following conditions on f_{ij}: for all $1 \le p, q \le n$,

$$f_{p,n+q} = -f_{q,n+p}, \quad f_{n+p,n+q} = f_{q,p}, \quad f_{n+p,q} = -f_{n+q,p}.$$

In other words, the matrix of f is necessarily of the form $(f_{ij}) = \begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}$ where B and C are $n \times n$ antisymmetric matrices. Hence the corresponding r-matrix α can be written as

$$\alpha = \begin{pmatrix} -B & A \\ -{}^tA & C \end{pmatrix}.$$

Reversing the arguments, we also get the converse. □

Remark 3.4. If there exists $1 \le k \le n$ such that $f_{n+k,k} \neq 0$ (or $f_{k-n,k} \neq 0$ for $n+1 \le k \le 2n$) then the central structure defined by f on $\mathcal{H}_{2n+1}$ is not exact.

We denote by I_p the identity matrix $p \times p$.

Example 3.5. The exact central structures on $\mathcal{H}_3$ are all of type $a\,\mathrm{I}_2$, where $a \in \mathbb{R}$. The corresponding r-matrix is given by $r = aX_1 \wedge X_2$.

4. Equivalence of Central Structures on $\mathcal{H}_{2n+1}$

Two central structures ε_f and ε_g on $\mathcal{H}_{2n+1}$, defined by f and g in $\mathrm{End}(\mathbb{R}^{2n})$ respectively, will be called equivalent if they define equivalent Lie bialgebra structures, i.e., if there exists a Lie algebra automorphism φ of $\mathcal{H}_{2n+1}$, say $\varphi \in \mathrm{Aut}(\mathcal{H}_{2n+1})$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_{2n+1} & \xrightarrow{\varphi} & \mathcal{H}_{2n+1} \\ {\scriptstyle\varepsilon_f}\downarrow & & \downarrow{\scriptstyle\varepsilon_g} \\ \wedge^2\mathcal{H}_{2n+1} & \xrightarrow{\varphi\otimes\varphi} & \wedge^2\mathcal{H}_{2n+1} \end{array}$$

We define the "extended" symplectic group of $\mathbb{R}^{2n}$ by

$$S(2n,\mathbb{R}) := \{A \in GL(2n,\mathbb{R}) \mid \exists s \in \mathbb{R}^* : \ {}^tAJA = sJ\} \text{ where } J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

Proposition 4.1. *Let f and g be two endomorphisms of $\mathbb{R}^{2n}$ with the matrixes M_f and M_g in the canonical basis of $\mathbb{R}^{2n}$, respectively. The central structures ε_f and ε_g on $\mathcal{H}_{2n+1}$ are equivalent if and only if there exists $A \in S(2n,\mathbb{R})$ such that $M_g = sAM_fA^{-1}$, where s is defined by ${}^tAJA = sJ$.*

The proof is an immediate consequence of the following realization of $\mathrm{Aut}(\mathcal{H}_{2n+1})$.

Lemma 4.2. *The automorphisms group of $\mathcal{H}_{2n+1}$ is given by*

$$G = \left\{ \begin{pmatrix} A & 0 \\ v & s \end{pmatrix} \in GL(2n+1,\mathbb{R}) \;\middle|\; \begin{array}{c} {}^tAJA = sJ, \text{ with } s \in \mathbb{R}^*, \text{ and} \\ v = (v_1, \cdots, v_{2n}) \in \mathbb{R}^{2n} \text{ arbitrary} \end{array} \right\}.$$

Proof. We distinguish the following classes of automorphisms of $\mathcal{H}_{2n+1}$ that we represent by their matrices in the canonical basis $((X_k)_{k=1}^{2n}, Z)$.

i) The symplectic automorphisms: $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ where A is a $2n \times 2n$ symplectic matrix, i.e., ${}^tAJA = J$.

ii) The inner automorphisms: $\begin{pmatrix} \mathrm{I}_{2n} & 0 \\ v & 1 \end{pmatrix}$ where $v \in \mathbb{R}^{2n}$.

iii) The dilatations: $\begin{pmatrix} r\,\mathrm{I}_{2n} & 0 \\ 0 & r^2 \end{pmatrix}$ where $r > 0$.

iv) The inversion: $\begin{pmatrix} 0 & \mathrm{I}_n & 0 \\ \mathrm{I}_n & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Every automorphism of $\mathcal{H}_{2n+1}$ can be written as a product of these automorphisms. It is easy to see that $\mathrm{Aut}(\mathcal{H}_{2n+1}) \subset G$. Reciprocally, consider an element $\sigma = \begin{pmatrix} A & 0 \\ v & s \end{pmatrix}$ of G. By multiplying this matrix by an inversion, we can assume that $s > 0$. By multiplying σ by the dilatation of coefficient $\frac{1}{\sqrt{s}}$ followed by a convenient inner automorphism, we obtain a matrix of the form $\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$, where B is a $2n \times 2n$ symplectic matrix. Thus σ is an automorphism of $\mathcal{H}_{2n+1}$, i.e., $G \subset \mathrm{Aut}(\mathcal{H}_{2n+1})$ hence $G = \mathrm{Aut}(\mathcal{H}_{2n+1})$. □

5. Conclusion

This work is motivated by the open question about the classification of all Lie bialgebra structures on a given (nilpotent) Lie algebra.

Every nilpotent Lie algebra can be obtained by successives central extensions from an abelian Lie algebra. Our hope was to use the notion of Lie bialgebra central extensions in order to classify all Lie bialgebra structures on a fixed nilpotent Lie algebra.

Our test Lie bilagebra was the Heisenberg algebra where all Lie bialgebra structures are known. Let us recall this result in terms of our previous notations.

Theorem 5.1. [4] *Each Lie bialgebra structure* $\varepsilon : \mathcal{H}_{2n+1} \to \wedge^2 \mathcal{H}_{2n+1}$ *on* $\mathcal{H}_{2n+1}$ *is one of the following two forms:*

(i) $\varepsilon(Z) = 0$ *and for all* $X \in \mathbb{R}^{2n}$, $\varepsilon(X) = Z \wedge f(X)$ *with* $f \in \mathrm{End}(\mathbb{R}^{2n})$.

(ii) $\varepsilon(Z) = Z \wedge A$ *and* $\forall X \in \mathbb{R}^{2n}$, $\varepsilon(X) = Z \wedge N_A(X) + \frac{1}{2}(X \wedge A + \gamma(X, A)\widetilde{\gamma}^{-1})$ *with* $A \in \mathbb{R}^{2n} - \{0\}$ *and* N_A *is the endomorphism of* $\mathbb{R}^{2n}$ *given by* $N_A = \gamma(A, U)\,\mathrm{I}_{2n} + U \otimes \widetilde{\gamma}(A) - A \otimes \widetilde{\gamma}(U) + \lambda A \otimes \widetilde{\gamma}(A)$, $U \in \mathbb{R}^{2n}$, $\lambda \in \mathbb{R}$.

Following our study, the first family (i) is the central structures on $\mathcal{H}_{2n+1}$ realized by the notion of Lie bialgebra central extensions. The second family (ii) cannot be obtained by this notion: One can see that for central extensions $\varepsilon(Z)$ vanishes necessarily. This is not the case for the second family (ii).

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Université des Sciences et Technologies de Lille UFR de Mathématiques-URA au CNRS D751 59655 Villeneuve d'Ascq Cedex; France.
benayed@gat.univ-lille1.fr

Université Mohammed V. Faculté des Sciences Département de Mathématiques et Informatique B.P 1014. Rabat; Maroc.
souidi@maghrebnet.net.ma

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