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Boundary Stabilization of a Hyperbolic Equation with Viscosity

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Abstract. This paper is concerned with the solvability and uniform stability of a hyperbolic equation with spatially varying coefficients of viscosity and elasticity and boundary damping.

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1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^n$ with C^2 boundary Γ and let (Γ_0, Γ_1) be a partition of Γ;both parts with positive measure.

We consider the following hyperbolic problem:

$$(1.1)\quad \left| \begin{array}{l} \dfrac{\partial^2 y}{\partial t^2} - \nabla \cdot (a_{ij}(x)\nabla y) - \dfrac{\partial}{\partial t}\nabla \cdot (b_{ij}(x)\nabla y) = f \quad \text{in} \quad Q = \Omega \times (0,\infty) \\ y = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty) \\ \dfrac{\partial y}{\partial \nu_a} + \dfrac{\partial}{\partial t}\dfrac{\partial y}{\partial \nu_b} + \beta(x)\left(\dfrac{\partial y}{\partial t} - g\right) = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty) \\ y(0) = y^0; \quad \dfrac{\partial y}{\partial t}(0) = y^1 \quad \text{in} \quad \Omega, \end{array} \right.$$

where $\frac{\partial y}{\partial \nu_a}$ (resp. $\frac{\partial y}{\partial \nu_b}$) is the outer cornormal derivative with respect to the matrix (a_{ij}) (resp. (b_{ij})) defined by

$$\frac{\partial y}{\partial \nu_a} = \nu_i\, a_{ij}\, \frac{\partial y}{\partial x_j}$$

and $\nu = (\nu_1, \cdots, \nu_n)$ denotes the exterior unit normal at Γ.

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In physical terms the entries a_{ij} and b_{ij} are related to coefficients of elasticity and viscosity, respectively. Without viscosity, i.e., if $b_{ij}=0$ and assuming that $a_{ij}=1$, $f=0$ and $g=0$, the problem (1.1) was studied by many authors: See J. P. Quinn and D. L. Russell [10], G. Chen [2, 3, 4], J. E. Lagnese [7, 8] and V. Komornik and E. Zuazua [6]. And when $\beta=0$, an asymptotic regularization procedure is proved by G. C. Hsiao and J. Sprekels [5]. Inspired by the above works we show solvability of strong and weak solutions to problem (1.1), and obtain boundary stabilization.

To obtain the existence of solutions we make use of Galerkin's aproximation. However, as we are also interested in srong solutions we have some technical difficulties wich lead us to transform the problem (1.1) into an equivalent one with zero initial data.

Stability problems with nonhomogeneous conditions require a special treatement because we don't have any information about the influence of the inner products $(f(t),y'(t))_{L^2(\Omega)}$ and $(g(t),y'(t))_{L^2(\Gamma_0)}$ on the energy

$$E(t)=\frac{1}{2}\int_\Omega |y'(x,t)|^2dx+\frac{1}{2}\int_\Omega a_{ij}(x)\frac{\partial y}{\partial x_i}(x,t)\frac{\partial y}{\partial x_j}(x,t)\,dx \tag{1.2}$$

or about the sign of its derivative $E'(t)$.

To obtain the uniform decay we use the perturbed energy method. Our paper is organized as follows. In Section 2 we give notations and state our main result. In Section 3 we prove solvability of strong and weak solutions of (1.1) while in Section 4 we obtain the boundary stabilization of solutions from Section 3.

2. Notations and Main Result

We define

$$(u,v)=\int_\Omega u(x)v(x)\,dx;\quad |u|^2=\int_\Omega |u(x)|^2\,dx,$$

$$(u,v)_{\Gamma_0}=\int_{\Gamma_0} u(x)v(x)\,dx;\quad |u|^2_{\Gamma_0}=\int_{\Gamma_0}|u(x)|^2dx$$

and let V be

$$V=\{v\in H^1(\Omega)\;;\;v=0\quad\text{on}\quad\Gamma_1\}$$

which, equipped with the topology $|\nabla\cdot|$ is a Hilbert subspace of $H^1(\Omega)$.

In order to establish our main result, we make the following assumptions on the coefficients:

$$a_{ij},b_{ij}\in C^1(\overline{\Omega}). \tag{2.1}$$

There exist positive constants a_0 and b_0 such that

$$a_{ij}=a_{ji},\quad\text{and for}\quad \xi\in\mathbf{R}^n,\ \sum_{i,j=1}^n a_{ij}\xi_i\xi_j\ge a_0|\xi|^2 \tag{2.2}$$

$$b_{ij}=b_{ji},\quad\text{and for}\quad \xi\in\mathbf{R}^n,\ \sum_{i,j=1}^n b_{ij}\xi_i\xi_j\ge b_0|\xi|^2, \tag{2.3}$$

$$\begin{gathered}\beta\in L^\infty(\Omega);\quad \beta(x)\ge 0\quad\text{a.e. on}\quad\Gamma_0\,,\ \ \beta(x)=0\ \ \forall x\in\overline{\Gamma_0}\cap\overline{\Gamma_1}\\ \text{and}\quad \beta(x)\to 0\quad\text{whenever } x \text{ tends to a point}\quad x\in\overline{\Gamma_0}\cap\overline{\Gamma_1}.\end{gathered} \tag{2.4}$$

This choice of the function β was done in order to avoid eventual singularities.

Defining

$$a(u,v)=\sum_{i,j=1}^{n}\int_{\Omega}a_{i,j}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx,\quad \forall u,v\in V \tag{2.5}$$

$$b(u,v)=\sum_{i,j=1}^{n}\int_{\Omega}b_{i,j}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx,\quad \forall u,v\in V \tag{2.6}$$

from (2.1), (2.2) and (2.3) there exist positive constants a_0, a_1, b_0 and b_1 such that

$$a_0|\nabla u|^2 \le a(u,u) \le a_1|\nabla u|^2,\quad \forall u\in V \tag{2.7}$$

$$b_0|\nabla u|^2 \le b(u,u) \le b_1|\nabla u|^2,\quad \forall u\in V. \tag{2.8}$$

Now, we are in position to state our main result.

Theorem 2.1. *Let*

$$\{y^0, y^1, f, g\} \in V\times L^2(\Omega)\times L^2(0,\infty;L^2(\Omega))\times H^1(0,\infty;L^2(\Gamma_0)).$$

Under the assumptions (2.1)-(2.3) *and assuming that* $g(0)=0$, *the problem* (1.1) *possesses a unique weak solution* $y:\Omega\times(0,\infty)\to\mathbf{R}$ *such that*

$$y\in L^\infty(0,\infty;V),\ \ y'\in L^\infty(0,\infty;L^2(\Omega)). \tag{2.9}$$

Moreover, provided that for large t, *the inequality*

$$\int_0^t \exp(\frac{\varepsilon}{2}s)\left(|f(s)|^2+|\sqrt{\beta}g(s)|^2_{\Gamma_0}\right)ds \le \alpha t^r \tag{2.10}$$

holds for some positive constants ε, α, *and* r, *we obtain the following energy decay*

$$E(t)\le C\exp(-\frac{\varepsilon}{2}t),\quad \forall t\ge 0 \quad \textit{and} \quad \forall\varepsilon\in(0,\varepsilon_0]$$

where C *and* ε_0 *are positive constants.*

Remark 1. The hypothesis (2.10) means that the map

$$t\longmapsto \int_0^t \exp(\frac{\varepsilon}{2}s)\left(|f(s)|^2+|\sqrt{\beta}g(s)|^2_{\Gamma_0}\right)ds$$

must be bounded by a polinomial $P(t)$.

3. Solvability of strong and weak solutions

In this section we are going to prove the existence of strong solutions of problem (1.1). Using density arguments we conclude the same for weak solutions. The existence of solutions may be proven either by the Galerkin method or using semigroup arguments. We employ the Galerkin method.

We define the Hilbert space

$$H=\{u\in V\,;\,Au, Bu\in L^2(\Omega)\} \tag{3.1}$$

where A and B are the operators defined by

$$A=-\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)\quad \text{and}\quad B=-\frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

and H is endowed by the natural inner product

$$(u,v)_H=(u,v)_V+(Au,Av)+(Bu,Bv).$$

Let us consider

$$y^0, y^1 \in H \tag{3.2}$$

satisfying the compatibility condition

$$\frac{\partial y^0}{\partial \nu_a} + \frac{\partial y^1}{\partial \nu_b} + \beta(x)\left(y^1 - g(0)\right) = 0. \tag{3.3}$$

In addition, let us assume that

$$f \in H^1(0,\infty; L^2(\Omega)), \quad g \in H^2(0,\infty; L^2(\Gamma_0)). \tag{3.4}$$

The variational formulation associated with problem (1.1) is given by

$$(y''(t), w) + a(y(t), w) + b(y(t), w) + (\beta y'(t), w)_{\Gamma_0} \\ = (f(t), w) + (\beta g(t), w)_{\Gamma_0}; \quad \forall w \in V.$$

In order to obtain strong solutions and since we can not use a 'special basis' (for instance, one formed by eigenfunctions) because of the boundary condition $(\beta y'(t), w)_{\Gamma_0}$, we need to derive the above expression with respect to t. But it lead us to technical difficulties when we estimate $y''(0)$. To solve this problem we transform the boundary value problem (1.1) into an equivalent one with null initial data. In fact, considering the change of variables

$$v(x,t) = y(x,t) - \phi(x,t) \tag{3.5}$$

where

$$\phi(x,t) = y^0(x) + t y^1(x), \quad (x,t) \in \Omega \times [0,\infty)$$

we obtain the equivalent problem for v :

$$\left| \begin{array}{l} v'' - \nabla\cdot(a_{ij}(x)\nabla v) - \nabla\cdot(b_{ij}(x)\nabla v') = F \quad \text{in} \quad Q \\ v = 0 \quad \text{on} \quad \Sigma_1 \\ \dfrac{\partial v}{\partial \nu_a} + \dfrac{\partial v'}{\partial \nu_b} + \beta v' = G \quad \text{on} \quad \Sigma_1 \\ v(0) = v'(0) = 0 \end{array} \right. \tag{3.6}$$

where

$$F = f + \nabla\cdot(a_{ij}(x)\nabla\phi) + \nabla\cdot(b_{ij}(x)\nabla\phi') \tag{3.7}$$

$$G = \beta g - \frac{\partial \phi}{\partial \nu_a} - \frac{\partial \phi'}{\partial \nu_b} - \beta\phi'. \tag{3.8}$$

If v is a solution of (3.6) in [0,T], then $y = v + \phi$ is a solution of (1.1) in the same interval. However, after two estimates we are going to prove later, we have that

$$|Av(t)|^2 + |\nabla v'(t)|^2 \le C(T), \ \ \forall t \in [0,T] \ \text{ and } \ \forall T > 0. \tag{3.9}$$

Thus, from (3.4) and (3.5) we have the same estimate obtained in (3.9) for the solution y. So, we can extend y to the whole interval $[0,\infty)$ using the standard argument

$$T_{max} = \infty \quad \text{or if} \quad T_{max} < \infty \quad \text{then} \quad \lim_{t\to T_{max}} \left(|Av(t)|^2 + |\nabla v'(t)|^2\right) = \infty.$$

Hence, it is sufficient to prove that (3.6) has a solution in [0,T], which will be done by the Galerkin method.

We represent by (ω_ν) a basis in H, which is orthonormal in $L^2(\Omega)$, by V_m the subspace of H generated by the m-first vectors $\omega_1, \cdots, \omega_m$ and define

$$v_m(t) = \sum_{i=1}^{m} g_{im}(t)\omega_i \tag{3.10}$$

where v_m(t) is the solution of the folowing Cauchy problem:

$$\begin{aligned}(v_m''(t), \omega_j) + a(v_m(t), \omega_j) + b(v_m'(t), \omega_j) + (\beta v_m'(t), \omega_j)_{\Gamma_0} \\ = (F(t), \omega_j) + (G(t), \omega_j)_{\Gamma_0}\,;\end{aligned} \tag{3.11}$$

$$v_m(0) = v_m'(0) = 0; \quad j = 1, \cdots, m.$$

The aproximate system is a normal one of ordinary differential equations. It has a solution in $[0, t_m)$. The extension of the solution on the whole interval [0,T] is a consequence of the first estimate we are going to obtain below.

3.1. **A Priori Estimates.**

3.1.1. THE FIRST ESTIMATE. Multiplying both sides of (3.11) by $g_{jm}'(t)$, summing over $1 \le j \le m$ and considering (2.2), we obtain

(3.12)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{|v_m'(t)|^2 + a(v_m(t), v_m(t))\right\} + b(v_m'(t), v_m'(t)) + \left|\sqrt{\beta}v_m'(t)\right|_{\Gamma_0}^2 \\ &\quad= (F(t), v_m'(t)) + \frac{d}{dt}(G(t), v_m(t))_{\Gamma_0} - (G'(t), v_m(t))_{\Gamma_0} \\ &\quad\le \frac{1}{2}|F(t)|^2 + \frac{1}{2}|v_m'(t)|^2 + \frac{d}{dt}(G(t), v_m(t))_{\Gamma_0} + \frac{C_0^2}{2}|G'(t)|_{\Gamma_0}^2 + \frac{1}{2}|\nabla v_m(t)|^2\end{aligned}$$

where C_0 is a positive constant such that $|v|_{\Gamma_0} \le C_0 |\nabla v|$, $\forall v \in V$.

Integrating (3.12) over (0,t) $0 < t < t_m$, taking into consideration (2.7) and (2.8) and noting that $v_m(0) = v_m'(0) = 0$, we get

$$\begin{aligned}&\frac{1}{2}|v_m'(t)|^2 + \frac{a_0}{2}|\nabla v_m(t)|^2 + b_0\int_0^t |\nabla v_m'(s)|^2\,ds + \int_0^t \left|\sqrt{\beta}v_m'(s)\right|_{\Gamma_0}^2 ds \\ &\qquad\le \frac{1}{2}\|F\|^2_{L^2(0,T;L^2(\Omega))} + \frac{C_0^2}{2}\|G'\|^2_{L^2(0,T;L^2(\Gamma_0))} + (G(t), v_m(t))_{\Gamma_0} \\ &\qquad\quad + \frac{1}{2}\int_0^t \left\{|v_m'(s)|^2 + |\nabla v_m(s)|^2\right\} ds.\end{aligned} \tag{3.13}$$

On the other hand, for ar arbitrary $\eta > 0$ we have

$$(G(t), v_m(t))_{\Gamma_0} \le \frac{C_0^2}{4\eta}|G(t)|_{\Gamma_0}^2 + \eta\,|\nabla v_m(t)|^2\,. \tag{3.14}$$

Combining (3.13), (3.14) and choosing $\eta > 0$ small enough, we obtain the first estimate

$$|v_m'(t)|^2 + |\nabla v_m(t)|^2 + \int_0^t |\nabla v_m'(s)|^2\,ds + \int_0^t \left|\sqrt{\beta}v_m'(s)\right|_{\Gamma_0}^2 ds \le L_1 \tag{3.15}$$

where L_1 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0, T]$.

3.1.2. The Second Estimate. First of all we are going to estimate $v''_m(0)$ in $L^2(\Omega)$ norm. Taking $t=0$ in (3.11) and taking into account that $v_m(0)=v'_m(0)=0$, it follows that

$$(v''_m(0),\omega_j)=(F(0),\omega_j)+(G(0),\omega_j)_{\Gamma_0}\,. \tag{3.16}$$

From (3.3) and (3.8) we have that $G(0)=0$, and from (3.16) we conclude

$$|v''_m(0)|^2=(F(0),v''_m(0))$$

which implies that

$$|v''_m(0)|\le L;\quad \forall m\in \mathbf{N} \tag{3.17}$$

where L is a positive constant independent of $m\in\mathbf{N}$.

Now, taking the derivative of (3.11) with respect to t, we can write

$$\begin{aligned}(v'''_m(t),\omega_j)+a(v'_m(t),\omega_j)+b(v''_m(t),\omega_j)+(\beta v''_m(t),\omega_j)_{\Gamma_0}&\\ =(F'(t),\omega_j)+(G'(t),\omega_j)_{\Gamma_0}\,.&\end{aligned} \tag{3.18}$$

Multiplying both sides of (3.18) by $g''_{jm}(t)$ and summing over $1\le j\le m$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{|v''_m(t)|^2+a(v'_m(t),v'_m(t))\right\}+b(v''_m(t),v''_m(t))+\left|\sqrt{\beta}v''_m(t)\right|^2_{\Gamma_0}&\\ =(F'(t),v''_m(t))+\frac{d}{dt}(G'(t),v'_m(t))_{\Gamma_0}-(G''(t),v'_m(t))_{\Gamma_0}\,.&\end{aligned} \tag{3.19}$$

Using arguments analogous to those considered in the first estimate, observing that $v'_m(0)=0$, and taking into account (3.17), from (3.19) we obtain the second estimate

$$|v''_m(t)|^2+|\nabla v'_m(t)|^2+\int_0^t|\nabla v''_m(s)|^2\,ds+\int_0^t\left|\sqrt{\beta}v''_m(s)\right|^2_{\Gamma_0}ds\le L_2 \tag{3.20}$$

where L_2 is a positive constant independent of $m\in\mathbf{N}$ and $t\in[0,T]$.

Due to estimates (3.15) and (3.20) we can extract a subsequence $\{v_\mu\}$ of $\{v_m\}$ such that

$$v_\mu \rightharpoonup v \quad \text{weak star in} \quad L^\infty(0,T;V) \tag{3.21}$$

$$v'_\mu \rightharpoonup v' \quad \text{weak star in} \quad L^\infty(0,T;V) \tag{3.22}$$

$$v''_\mu \rightharpoonup v'' \quad \text{weak in} \quad L^2(0,T;V) \tag{3.23}$$

$$v''_\mu \rightharpoonup v'' \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)) \tag{3.24}$$

$$\beta v'_\mu \rightharpoonup \beta v' \quad \text{weak in} \quad L^2(0,T;L^2(\Gamma_0)). \tag{3.25}$$

The above convergences are sufficient to pass to the limit.

3.2. **Uniqueness.** Supose we have two solutions y and $\hat{y}$ of problem (1.1). Then $z=y-\hat{y}$ satisfies

$$\begin{gathered}(z''(t),w)+a(z(t),w)+b(z'(t),w)+(\beta z'(t),w)_{\Gamma_0}=0,\ \forall w\in V\\ z(0)=z'(0)=0.\end{gathered} \tag{3.26}$$

Taking $w = 2z'(t)$ in (3.26) we obtain

$$\frac{d}{dt}\left\{|z'(t)|^2 + a(z(t), z(t))\right\} + 2b(z'(t), z'(t)) + 2\left|\sqrt{\beta}z'(t)\right|^2_{\Gamma_0} = 0. \tag{3.27}$$

Integrating (3.27) over (0,t) we obtain

$$|z'(t)|^2 + a_0\,|\nabla z(t)|^2 + 2b_0\int_0^t |\nabla z'(s)|^2\,ds + 2\int_0^t \left|\sqrt{\beta}z'(s)\right|^2_{\Gamma_0} ds = 0$$

which implies that $|\nabla z(t)| = |z'(t)| = 0$. This completes the proof.

3.3. **Solvability of weak solutions.** We have just proved the existence of solutions of the problem (1.1) when y^0 and y^1 are smooth. By density arguments we conclude the same for weak solutions. However, the principal difficulty is due to the existence of a sequence of initial data which satisfies the hypothesis of compatibility (3.3). For this end and since $g(0) = 0$, given $\{y^0, y^1\} \in V \times L^2(\Omega)$ it is sufficient to consider

$$y^0_\mu \in D(A) = \left\{u \in V\,;\, Au \in L^2(\Omega) \quad \text{and} \quad \frac{\partial u}{\partial \nu_a} = 0 \quad \text{on} \quad \Gamma_0\right\}$$

such that

$$y^0_\mu \to y^0 \quad \text{in} \quad V$$

and

$$y^1_\mu \in D(B) \cap H^1_0(\Omega) \quad \text{such that} \quad y^1_\mu \to y^1 \quad \text{in} \quad L^2(\Omega).$$

The uniqueness of weak solutions requires a regularization procedure and can be obtained using the standard method of Visik-Ladyshenskaya, cf. J. L. Lions [9] Chapter 3, Section 8.2.

4. Asymptotic Behaviour

In this section we obtain the uniform decay of the energy given in (1.2) for strong solutions, since the same occurs for weak solutions using standard density arguments.

The derivative of the energy given by (1.2) is

$$E'(t) = -b(y'(t), y'(t)) - \left|\sqrt{\beta}y'(t)\right|^2_{\Gamma_0} + (f(t), y'(t)) + (g(t), y'(t))_{\Gamma_0}\,. \tag{4.1}$$

Let λ and μ be positive constants such that

$$|v|^2 \le \lambda |\nabla v|^2 \qquad \forall v \in V \tag{4.2}$$

and

$$\left|\sqrt{\beta}v\right|^2_{\Gamma_0} \le \mu |\nabla v|^2;\ \ \forall v \in V. \tag{4.3}$$

For an arbitrary $\varepsilon > 0$ we define the perturbed energy

$$E_\varepsilon(t) = E(t) + \varepsilon\psi(t) \tag{4.4}$$

where

$$\psi(t) = \int_\Omega y' y\,dx. \tag{4.5}$$

Proposition 4.1. *There exists* $C_1 > 0$ *such that*

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon C_1 E(t), \ \forall t \geq 0 \ \ \textit{and} \ \ \forall \varepsilon > 0.$$

Proof. From (2.7), (4.2) and (4.5) we obtain

$$|\psi(t)| \leq \frac{1}{2}|y'|^2 + \frac{1}{2}\lambda\, a_0^{-1} a(y,y) \leq \left(\lambda a_0^{-1} + 1\right) E(t). \tag{4.6}$$

If we define $C_1 = \lambda a_0^{-1} + 1$,then from (4.4) and (4.6) we can write

$$|E_\varepsilon(t) - E(t)| = \varepsilon|\psi(t)| \leq \varepsilon C_1 E(t).$$

This concludes the proof. □

Proposition 4.2. *There exist* $C_2 = C_2(\varepsilon)$ *and* ε_1 *positive constants such that*

$$E_\varepsilon'(t) \leq -\varepsilon E(t) + C_2\left(|f(t)|^2 + |\sqrt{\beta} g(t)|^2_{\Gamma_0}\right); \quad \forall t \geq 0 \quad \textit{and} \quad \forall \varepsilon \in (0, \varepsilon_1].$$

Proof. First of all we must estimate $\psi'(t)$ in terms of $E(t)$. Taking the derivative of $\psi(t)$ given in (4.5) and replacing y'' by $\nabla \cdot (a_{ij}(x)\nabla y) + \nabla \cdot (b_{ij}\nabla y') + f$ in the expression obtained it follows that

$$\psi'(t) = \int_\Omega \nabla \cdot (a_{ij}(x)\nabla y)\, y\, dx + \int_\Omega \nabla \cdot (b_{ij}\nabla y')\, y\, dx + \int_\Omega f y\, dx + \int_\Omega |y'|^2 dx. \tag{4.7}$$

On the other hand, from Gauss' theorem and taking into account that

$$\frac{\partial y}{\partial \nu_a} + \frac{\partial y'}{\partial \nu_b} = \beta(g - y') \quad \text{on} \quad \Gamma_0$$

we obtain

$$\begin{aligned} &\int_\Omega \nabla \cdot (a_{ij}(x)\nabla y)\, y\, dx + \int_\Omega \nabla \cdot (b_{ij}\nabla y')\, y\, dx \\ &\qquad = -a(y(t), y(t)) - b(y'(t), y(t)) - \int_{\Gamma_0} \beta y' y\, d\Gamma + \int_{\Gamma_0} \beta g y\, d\Gamma. \end{aligned} \tag{4.8}$$

Replacing (4.8) in (4.7); adding and subtracting the term $\int_\Omega |y'|^2 dx$ in (4.7) we get

$$\psi'(t) = -2E(t) - b(y'(t), y(t)) + 2\int_\Omega |y'|^2 dx - \int_{\Gamma_0} \beta y' y\, d\Gamma + \int_\Omega f y\, dx + \int_{\Gamma_0} \beta g y\, d\Gamma. \tag{4.9}$$

Now, from (2.7), (4.2), (4.3) and (4.9), using for an arbitrary $\eta > 0$ the inequality $ab \leq \frac{a^2}{4\eta} + \eta b^2$ we can write

$$\begin{aligned} \psi'(t) \leq & -(2 - 8\eta)E(t) + \left(2\lambda + \frac{||b||^2 a_0^{-1}}{4\eta}\right)|\nabla y'(t)|^2 \\ & + \frac{\mu a_0^{-1}}{4\eta}\left|\sqrt{\beta} y'(t)\right|^2_{\Gamma_0} + \frac{a_0^{-1}\lambda}{4\eta}\left(|f(t)|^2 + \left|\sqrt{\beta} g(t)\right|^2_{\Gamma_0}\right) \end{aligned} \tag{4.10}$$

where

$$||b|| = \sum_{i,j=1}^n ||b_{ij}||_{L^\infty(\Omega)}.$$

Choosing $\eta = \frac{1}{8}$ from (4.10) we have

(4.11)
$$\psi'(t) \leq -E(t) + M_1 \left|\nabla y'(t)\right|^2 + M_2 \left|\sqrt{\beta} y'(t)\right|_{\Gamma_0}^2 + M_3 \left(|f(t)|^2 + \left|\sqrt{\beta} g(t)\right|_{\Gamma_0}^2\right)$$

where
$$M_1 = 2\left(\lambda + ||b||^2 a_0^{-1}\right), \quad M_2 = 2\mu a_0^{-1} \quad \text{and} \quad M_3 = 2a_0^{-1}\lambda.$$

Thus, combining (2.8), (4.1), (4.4) and (4.11), we conclude
$$\begin{aligned} E'_\varepsilon(t) &= E'(t) + \varepsilon\psi'(t) \\ &\leq -b_0 \left|\nabla y'(t)\right|^2 - \left|\sqrt{\beta} y'(t)\right|_{\Gamma_0}^2 + (f(t), y'(t)) + (\beta g(t), y'(t))_{\Gamma_0} - \varepsilon E(t) \\ &\quad + \varepsilon M_1 \left|\nabla y'(t)\right|^2 + \varepsilon M_2 \left|\sqrt{\beta} y'(t)\right|_{\Gamma_0}^2 + \varepsilon M_3 \left(|f(t)|^2 + \left|\sqrt{\beta} g(t)\right|_{\Gamma_0}^2\right) \end{aligned}$$

which implies

(4.12)
$$\begin{aligned} E'_\varepsilon(t) \leq &- (b_0 - \varepsilon(M_1+1)) \left|\nabla y'(t)\right|^2 - (1 - \varepsilon(M_2+1)) \left|\sqrt{\beta} y'(t)\right|_{\Gamma_0}^2 \\ &- \varepsilon E(t) + \left(\frac{\lambda}{4\varepsilon} + \varepsilon M_3\right) |f(t)|^2 + \left(\frac{1}{4\varepsilon} + \varepsilon M_3\right) |g(t)|_{\Gamma_0}^2. \end{aligned}$$

Defining
$$\varepsilon_1 = min\left\{\frac{b_0}{M_1+1}, \frac{1}{M_2+1}\right\}$$
and choosing $\varepsilon \in (0, \varepsilon_1]$ it follows that
$$E'_\varepsilon(t) \leq -\varepsilon E(t) + C_2(\varepsilon)\left(|f(t)|^2 + \left|\sqrt{\beta} g(t)\right|_{\Gamma_0}^2\right)$$
which concludes the proof. □

Proof of the exponential decay. We define
$$\varepsilon_0 = min\{\varepsilon_1, \frac{1}{2C_1}\}$$
and let us consider $\varepsilon \in (0, \varepsilon_0]$. From Proposition 4.1 we have

(4.13)
$$(1 - C_1\varepsilon)E(t) \leq E_\varepsilon(t) \leq (1 + C_1\varepsilon)E(t)$$

Since $\varepsilon \leq 1/2C_1$,

(4.14)
$$\frac{1}{2}E(t) \leq E_\varepsilon(t) \leq \frac{3}{2}E(t) \leq 2E(t); \;\; \forall t \geq 0$$

and therefore

(4.15)
$$-\varepsilon E(t) \leq -\frac{\varepsilon}{2}E_\varepsilon(t).$$

Hence, from (4.15) and considering Proposition 4.2 we obtain
$$E'_\varepsilon(t) \leq -\frac{\varepsilon}{2}E_\varepsilon(t) + C_2\left(|f(t)|^2 + |\sqrt{\beta} g(t)|_{\Gamma_0}^2\right).$$

Consequently
$$\frac{d}{dt}\left(E_\varepsilon(t)\exp(\frac{\varepsilon}{2}t)\right) \leq C_2\left(|f(t)|^2 + |\sqrt{\beta} g(t)|_{\Gamma_0}^2\right)\exp(\frac{\varepsilon}{2}t).$$

Integrating the above inequality over [0,t] we get

$$E_\varepsilon(t) \le \exp(-\frac{\varepsilon}{2}t)E_\varepsilon(0) + C_2\exp(-\frac{\varepsilon}{2}t)\int_0^t \exp(\frac{\varepsilon}{2}s)\left(|f(s)|^2 + |\sqrt{\beta}g(s)|^2_{\Gamma_0}\right)$$

and taking into consideration (4.14) we see that

$$E(t) \le \left(3E(0) + 2C_2\int_0^t \exp(\frac{\varepsilon}{2}s)\left(|f(t)|^2 + |\sqrt{\beta}g(t)|^2_{\Gamma_0}\right)\right)\exp(-\frac{\varepsilon}{2}t). \tag{4.16}$$

Combining (4.16) with the assumption (2.10) we prove the desired decay and finish the proof of Theorem 2.1. □

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