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On Metric Diophantine Approximation and Subsequence Ergodic Theory

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Abstract. Suppose k_n denotes either $\phi(n)$ or $\phi(p_n)$ $(n = 1, 2, \cdots)$ where the polynomial ϕ maps the natural numbers to themselves and p_k denotes the k^{th} rational prime. Let $(\frac{r_n}{q_n})_{n=1}^{\infty}$ denote the sequence of convergents to a real number x and define the the sequence of approximation constants $(\theta_n(x))_{n=1}^{\infty}$ by

$$\theta_n(x) = q_n^2 \left| x - \frac{r_n}{q_n} \right|. \qquad (n = 1, 2, \cdots)$$

In this paper we study the behaviour of the sequence $(\theta_{k_n}(x))_{n=1}^{\infty}$ for almost all x with respect to Lebesgue measure. In the special case where $k_n = n$ $(n = 1, 2, \cdots)$ these results are due to W. Bosma, H. Jager and F. Wiedijk.

Contents

1. Introduction

In this paper we study the behaviour of the regular continued fraction expansion of a real number

$$x = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 \ddots}}}},$$

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which is also written more compactly as $[c_0; c_1, c_2, \ldots]$. The terms $c_0, c_1, \ldots$ are called the partial quotients of the continued fraction expansion and the sequence of rational truncates

$$[c_0; c_1, \ldots, c_n] = \frac{p_n}{q_n}, \qquad (n = 1, 2, \ldots)$$

are called the convergents of the continued fraction expansion. More particularly recall the inequality

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}, \tag{1.1}$$

which is classical and well known [HW]. Clearly if for each natural number n we set

$$\theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right|,$$

then for each x the sequence $(\theta_n(x))_{n=1}^{\infty}$ lies in the interval $[0, 1]$. The distribution for almost all x with respect to Lebesgue measure of the sequence $(\theta_n(x))_{n=1}^{\infty}$ is studied in [BJW]. In this paper extending work in [BJW] we use ergodic theory to study some other functions of this sequence. In Section 2 we collect together some ergodic theoretic prerequisites. In Section 3 we state and prove our main result concerning the distribution of $(\theta_n(x))_{n=1}^{\infty}$ which refines the work in [BJW]. Finally in Section 4 the method of Section 3 is adapted to study some other sequences attached to the continued fraction expansion of x.

2. Basic Ergodic Theory

Here and throughout the rest of the paper by a dynamical system (X, β, μ, T) we mean a set X, together with a σ-algebra β of subsets of X, a probability measure μ on the measurable space (X, β) and a measurable self map T of X that is also measure preserving. By this we mean that if given an element A of β if we set $T^{-1}A = \{x \in X : Tx \in A\}$ then $\mu(A) = \mu(T^{-1}A)$. We say a dynamical system is ergodic if $T^{-1}A = A$ for some A in β means that $\mu(A)$ is either zero or one in value. We say the dynamical system (X, β, μ, T) is weak mixing (among other equivalent formulations [Wa]) if for each pair of sets A and B in β we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| = 0.$$

Weak mixing is a strictly stronger condition than ergodicity. A piece of terminology that is becoming increasingly standard is to call a sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ of non-negative integers L^p good universal if given any dynamical system (X, β, μ, T) and any function f in $L^p(X, \beta, \mu)$ it is true that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{k_n}x) = \ell_f(x),$$

exists almost everywhere with respect to the measure μ. Here and henceforth, for each real number y let $[y]$ denote the greatest integer less than y and let $\langle y \rangle = y - [y]$. The following theorem is a consequence of Theorem 2.3 in [Na2].

Theorem 2.1. *Suppose the sequence* $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *of non-negative integers is such that for each irrational number* α *the sequence* $(\langle k_n\alpha\rangle)_{n=1}^{\infty}$ *is uniformly distributed modulo one and that for a particular* p *greater or equal to one that* $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *is* L^p *good universal. Then if the dynamical system* (X, β, μ, T) *is weak mixing* $\ell_f(x) = \int_X f(t)d\mu(t)$ *almost everywhere with respect to* μ.

If k_n denotes either $\phi(n)$ or $\phi(p_n)$ where ϕ denotes any non-constant polynomial mapping the natural numbers to themselves and p_n denotes the n^{th} rational prime then $\mathbf{k}$ is L^p good universal for any p greater than one. See [Bo2] and [Na1] respectively for proofs, and the 1989 Ohio State Ph.D thesis of M. Wierdl for related results. The fact that for each irrational number α the sequence $(\langle k_n\alpha\rangle)_{n=1}^{\infty}$ is uniformly distributed modulo one in both instances are well known classical results. See [We] and [Rh] respectively. Other sequences are known by the author to satisfy the both hypotheses but these results have yet to appear in print [Na3].

We now consider the particular ergodic properties of the Gauss map, defined on $[0, 1]$ by

$$Tx = \left\langle \frac{1}{x} \right\rangle \ x \neq 0 \ ; \ T0 = 0.$$

Notice that $c_n(x) = c_{n-1}(Tx)$ $(n = 1, 2, \cdots)$. The dynamical system (X, β, μ, T) where X denotes $[0, 1]$, β is the σ-algebra of Borel sets on X, μ is the measure on (X, β) defined for any A in β by

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{dx}{x+1},$$

and T is the Gauss map is weak mixing. See [CFS] for details. The ergodic properties of the dynamical system (X, β, μ, T) are not quite enough to carry out this investigation. We also need ergodic theoretic information about its natural extention. In particular we need the following theorem from [INT]. See [CFS] for a definition of the natural extention and [S] for other general background.

Theorem 2.2. *Let* $\Omega = ([0, 1) \setminus \mathbf{Q}) \times [0, 1]$. *Now let* γ *be the* σ*-algebra of Borel subsets of* Ω *and let* ω *be the probability measure on the measurable space* (Ω, β) *defined by*

$$\omega(A) = \frac{1}{(\log 2)}\int_A \frac{dxdy}{(1+xy)^2}.$$

Also define the map

$$\mathcal{T}(x, y) = (Tx, \frac{1}{[\frac{1}{x}] + y}).$$

Then the map $\mathcal{T}$ *preserves the measure* ω *and the dynamical system* $(\Omega, \beta, \omega, \mathcal{T})$ *is weak mixing.*

Note that

$$\mathcal{T}^n(x, y) = (T^n x, [0; a_n, a_{n-1}, \cdots, a_2, a_1 + y]) \qquad (0 \leq y \leq 1, n = 1, 2, \cdots)$$

and in particular

$$\mathcal{T}^n(x, 0) = (T^n x, \frac{q_{n-1}}{q_n}).$$

3. Statistical Properties of the Sequence $(\theta_n(x))_{n=1}^{\infty}$

The main result of this paper is the following.

Theorem 3.1. *Suppose the sequence of integers* $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *satisfies the hypothesis of Theorem* 2.1. *Let the function* $F_1 : [0,1] \to [0,1]$ *be defined by* $F_1(z) = \frac{z}{\log 2}$ *on* $[0, \frac{1}{2}]$ *and* $F_1(z) = \frac{1}{\log 2}(1 - z + \log 2z)$ *on* $[\frac{1}{2}, 1]$. *Then*

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n : \theta_{k_j}(x) \le z\}| = F_1(z), \tag{3.2}$$

almost everywhere with respect to Lebesgue measure.

In the special case $k_n = n \quad (n = 1, 2, \dots)$ this result was conjectured by H. W. Lenstra Jr. and proved in [BJW].

Proof of Theorem 3.1. Denote by $\Omega(c)$ with $c \ge 1$ that part of Ω on or above the hyperbola $\frac{1}{x} + y = c$. In [K, p. 29] it is noted that

$$\theta_n(x) = \frac{1}{(\frac{1}{T^n x} + \frac{q_{n-1}}{q_n})} \qquad (n = 1, 2, \dots),$$

the statement $\theta_n(x) \le z$ for $z \in [0,1]$ is equivalent to the statement that $\mathcal{T}^n(x,0) \in \Omega(\frac{1}{z})$. It is also readily verified there exists an integer $n_0(\epsilon)$ such that for all n greater than $n_0(\epsilon)$ and all y in $[0,1]$ if

$$\mathcal{T}^n(x,y) \in \Omega(\frac{1}{z} + \epsilon)$$

then

$$\mathcal{T}^n(x,0) \in \Omega(\frac{1}{z}).$$

Also if

$$\mathcal{T}^n(x,0) \in \Omega(\frac{1}{z})$$

then

$$\mathcal{T}^n(x,y) \in \Omega(\frac{1}{z} - \epsilon).$$

From this it follows that for almost all (x,y) with respect to the measure μ we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n \ ; \ \mathcal{T}^{k_j}(x,y) \in \Omega(\frac{1}{z} + \epsilon)\}| \\
&\le \liminf_{n\to\infty} \frac{1}{n}|\{1 \le j \le n \ ; \ \mathcal{T}^{k_j}(x,0) \in \Omega(\frac{1}{z})\}| \\
&\le \limsup_{n\to\infty} \frac{1}{n}|\{1 \le j \le n \ ; \ \mathcal{T}^{k_j}(x,0) \in \Omega(\frac{1}{z})\}| \\
&\le \lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n \ ; \ \mathcal{T}^{k_j}(x,y) \in \Omega(\frac{1}{z} - \epsilon)\}|.
\end{aligned}$$

Using the fact that $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is L^p good universal, both limits exist and are $\mu(\Omega(\frac{1}{z} + \epsilon))$ and $\mu(\Omega(\frac{1}{z} - \epsilon))$ respectively. Since ϵ is arbitrary the limit (3.2) exists and is equal to $\mu(\Omega(\frac{1}{z}))$ for almost all x with respect to Lebesgue measure. We straightforwardly verify that $\mu(\Omega(\frac{1}{z})) = F(z)$. □

Corollary 3.3. *Suppose the sequence* $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *satisfies the hypothesis of Theorem* 2.1. *Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \theta_{k_j}(x) = \frac{1}{4\log 2},$$

almost everywhere with respect to Lebesgue measure.

Proof. This follows immediately from the fact that the first moment $\int_0^1 z dF_1(z)$ has the value $\frac{1}{4\log 2}$. □

4. Other Sequences Attached to the Regular Continued Fraction Expansion

Theorem 4.1. *Suppose* z *is in* $[0,1]$ *and for irrational* x *in* $(0,1)$ *set* $Q_n(x) = \frac{q_{n-1}(x)}{q_n(x)}$ *for each positive integer* n. *Suppose also that* $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *satisfies the hypothesis of Theorem* 2.1. *Then*

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n : Q_{k_j}(x) \le z\}| = F_2(z) = \frac{\log(1 + z)}{\log 2}$$

almost everywhere with respect to Lebesgue measure.

Proof. Using the fact that $\mathbf{k} = (k_n)_{n=1}^{\infty}$ satisfies the hypothesis of Theorem 2.1, we see that

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n : T^{k_j}(x) \le z\}| = \frac{1}{\log 2}\int_0^z \frac{dx}{1 + x} = \frac{\log(1 + z)}{\log 2}.$$

Now note that, for a set E in β if $\overline{E}$ denotes $\{(x,y) : (y,x) \in E\}$ then $\mu(E) = \mu(\overline{E})$ and so $(Q_{k_j}(x))_{n=1}^{\infty}$ is distributed identically to $(T^{k_j}x)_{j=1}^{\infty}$ and the theorem follows as a consequence. □

Theorem 4.2. *For irrational* x *in* $(0,1)$ *set*

$$r_n(x) = \frac{|x - \frac{p_n}{q_n}|}{|x - \frac{p_{n-1}}{q_{n-1}}|}. \qquad (n = 1,2,\dots)$$

Further for z *in* $[0,1]$ *let*

$$F_3(z) = \frac{1}{\log 2}(\log(1 + z) - \frac{z}{1 + z}\log z). \tag{4.3}$$

Suppose also that $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *satisfies the hypothesis of Theorem* 2.1. *Then*

$$\lim_{n\to\infty} \frac{1}{n}|\{1 \le j \le n : r_{k_j}(x) \le z\}| = F_3(z),$$

almost everywhere with respect to Lebesgue measure.

Proof. It follows from the fact that

$$x - \frac{p_n}{q_n} = \frac{(-1)^n T^n x}{q_n(q_n + q_{n-1}T^n x)} \tag{4.4}$$

[B, pp. 41–42] and the fact that

$$\frac{1}{T^{n-1}x} = a_n + T^n x$$

that $r_n(x) = \frac{q_{n-1}}{q_n}T^n x$. Arguing as in the proof of Theorem 3.1 we see that F_3 exists for almost all x and that for z in $[0,1]$ the value of $F_3(z)$ is equal to the μ measure of the part of Ω under the curve $xy = z$. A simple calculation shows that F_3 is given by (4.3) as specified. □

Corollary 4.5. *Suppose the sequence* $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *satisfies the hypothesis of Theorem* 2.1. *Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} r_{k_j}(x) = \frac{\pi^2}{12\log 2} - 1,$$

almost everywhere with respect to Lebesgue measure.

Proof. The limit is $\int_0^1 z dF_3(z)$. □

Another well known inequality is the following

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \qquad (n = 1, 2, \dots)$$

which motivates the following result.

Theorem 4.6. *For each irrational number* x *in* $(0,1)$ *define the function* $d_n(x)$ *for each natural number* n *by the identity*

$$\left|x - \frac{p_n}{q_n}\right| = \frac{d_n(x)}{q_n q_{n+1}}. \tag{4.7}$$

Suppose the sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ *satisfies the hypothesis of Theorem* 2.1. *Suppose also* F_4 *is defined on* $[0,1]$ *as* $F_4(z) = 0$ *if* z *is in* $[0, \frac{1}{2}]$ *and*

$$F_4(z) = \frac{1}{\log 2}(z\log z + (1 - z)\log(1 - z) + \log 2)$$

if z *is in* $[\frac{1}{2}, 1]$. *Then*

$$\lim_{n\to\infty}\frac{1}{n}|\{1 \le j \le n : d_{k_j}(x) \le z\}| = F_4(z),$$

almost everywhere with respect to Lebesgue measure.

Proof. From (4.4) and (4.7) we readily see that

$$d_n(x) = \frac{1}{1 + \frac{q_n}{q_{n-1}}T^{n+1}x}. \qquad (n = 1, 2, \dots)$$

Hence $F_4(z)$ equals the μ measure of the part of Ω above the curve $xy = \frac{1}{z} - 1$. Note that for $z \le \frac{1}{2}$ this is an empty set. □

Finally in this section we consider the inequality

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{2q_n q_{n-1}}. \qquad (n = 1, 2, \dots)$$

This is sharper than (1.1) whenever $c_n = 1$. That is for almost all x with frequency $2 - \frac{\log 3}{\log 2}$. See [B] for details. This motivates the following theorem.

Theorem 4.8. *For each irrational number x in $(0,1)$ define the function $D_n(x)$ for each natural number n by the identity*

$$\left|x - \frac{p_n}{q_n}\right| = \frac{D_n(x)}{q_n q_{n-1}}. \qquad (n = 1,2,\dots)$$

Suppose the sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ satisfies the hypothesis of Theorem 2.1. Suppose F_5 is defined on $[0,1]$

$$F_5(z) = \frac{1}{\log 2}(\log z - \frac{z}{2}\log z - \frac{2 - z}{2}\log(2 - z))$$

if z is in $[0,1]$. Then

$$\lim_{n\to\infty}\frac{1}{n}|\{1 \le j \le n : D_{k_j}(x) \le z\}| = F_5(z),$$

almost everywhere with respect to Lebesgue measure.

Proof. It is not difficult to verify that

$$D_n(x) = \frac{2}{(\frac{q_n}{q_{n-1}}\frac{1}{T^n x} + 1)}. \qquad (n = 1,2,\dots)$$

As earlier in the proof of Theorem 3.1 $F_5(z)$ denotes the μ measure of the part of Ω under the hyperbola $xy = \frac{z}{2 - z}$ when z is in $[0,1]$. □

Corollary 4.9. *Suppose the sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ satisfies the hypothesis of Theorem 2.1. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} D_{k_j}(x) = 1 - \frac{1}{2\log 2},$$

almost everywhere with respect to Lebesgue measure.

Proof. The limit is $\int_0^1 z dF_5(z) = 1 - \frac{1}{2\log 2}$. □

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