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# Multiple Rokhlin Tower Theorem: A Simple Proof

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Abstract. S. Alpern has proved that an invertible antiperiodic measurable measure preserving transformation of a Lebesgue probability space can be represented by $k$ towers of heights $n_1, \ldots, n_k$, with prescribed measures, provided that the heights have greatest common divisor 1. In this paper we give a simple proof of Alpern's theorem. It is elementary in the sense that it involves no limits and uses Kakutani's easy proof of Rokhlin's Lemma.

Contents



## 1. Alpern's Multiple Rokhlin Tower Theorem

In this paper we show that Kakutani's proof of Rokhlin's Lemma [Hal56] can be used to give a short, elementary proof of the following Multiple Rokhlin Tower Theorem of Alpern's [Alp79, Cor 2].

**Theorem 1.1** (Alpern). *For any $k \geq 2$, let $n_1, n_2, \ldots, n_k$ be relatively prime positive integers, and let $q_1, \ldots, q_k$ be positive numbers such that $n_1 q_1 + \cdots + n_k q_k = 1$. Then for any antiperiodic invertible measure preserving transformation $T$ of a Lebesgue probability space $(X, \Sigma, \mu)$, there exist sets $Q_i \in \Sigma, i = 1, \ldots, k$ with $\mu(Q_i) = q_i$ and such that $\{T^j(Q_i) : i = 1, \ldots, k,\ j = 0, \ldots, n_i - 1\}$ is a partition of $X$ (into $k$ columns of heights $n_1, \ldots, n_k$ and $\mu$-widths $q_1, \ldots, q_k$).*

Alpern applied this result to prove that in the space of measure preserving homeomorphisms of a compact connected manifold with the topology of uniform convergence, any measure theoretic property which is generic (dense $G_\delta$) in the weak topology on the space of invertible measure preserving transformations of the underlying probabilty measure space is also generic (dense $G_\delta$) in the uniform convergence topology in the group of measure preserving homeomorphisms. The latter result is a far-reaching generalization of the classical (1940) Oxtoby-Ulam Theorem [OU40], where it is proved that ergodicity is generic in the space of measure

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preserving homeomorphisms. Furthermore this contains Katok and Stepin's 1970 result [KS70] that weak mixing is generic for measure preserving homeomorphisms.

Very recently, N. Ormes [Orm] has used Alpern's Multiple Rokhlin Tower Theorem as one step in obtaining the following result about realizing an ergodic measure preserving system as a minimal homeomorphism of the Cantor set within a given topological orbit equivalence class: Let $Y$ be a Cantor set, $S$ a minimal homeomorphism of $Y$ and $\nu$ a uniquely ergodic $S$-invariant Borel probability measure. Let $T$ be an ergodic invertible measure preserving transformation of the Lebesgue probability space $(X, \Sigma, \mu)$. Then there is a topological realization $(S', \nu)$ of the ergodic system $(T, \mu)$, where $S'$ is a minimal homeomorphism of the Cantor set $Y$ strongly orbit equivalent to $S$ if and only if the finite rotations which are topological factors of $S$ are measurable factors of $T$.

Two homeomorphisms, $S$ and $S'$, of the Cantor set $Y$, are said to be strongly orbit equivalent if there is a homeomorphism $h : Y \to Y$ and integer valued maps $m, n : Y \to Z$ such that $hS^{m(x)}(x) = S'h(x), hS(x) = (S')^{n(x)}h(x)$ and $m$ and $n$ have no more than one point of discontinuity. Strong orbit equivalence of two minimal Cantor homeomorphisms has been identified in the work of Giordano, Putnam and Skau [GPS96, Theorem 2.2] as a necessary and sufficient condition for the isomorphism of the crossed product $C^*$-algebras $(C(Y) \times_S Z$, and $C(Y) \times_{S'} Z)$ associated with the minimal homeomorphisms .

Multiple Rokhlin Towers arise naturally in the study of minimal homeomorphisms $S$ of the Cantor set in the following manner. Given any clopen set $A$ in the Cantor set $Y$, consider $r_A : A \to N$, the first return time function to $A$ for the homeomorphism $S$, where $r_A(x)$ is the smallest positive integer such that $S^{r_A(x)}(x) \in A$. Then the continuity of this function implies there is a finite set of positive integers $n_1, \dots, n_k$ which is the range of $r_A$. This gives a Multiple Rokhlin Tower partition of $Y$ into clopen sets (into $k$ towers of heights $n_1, \dots, n_k$ over the base $A$). It is this tower for $S$ which Ormes "copies" for the ergodic $T$ using Alpern's theorem. While it need not be true that the $\gcd\{n_1, \dots, n_k\} = 1$, Ormes notes [Orm, Cor 4.2] that if $\gcd\{n_1, \dots, n_k\} = p$, then there is periodic clopen set for $S$ of period $p$. Thus to have a similar tower picture for $T$, there must also be a $T$-periodic set in $X$ of order $p$.

Extensions of Alpern's Multiple Rokhlin Tower Theorem to denumerably many columns and to nonsingular aperiodic transformations may be found in [Alp81] and [AP90]. In addition applications of these extensions to coding Markov chains and approximate conjugacy theorems can also be found in those papers.

## 2. Proof of Alpern's Theorem

**Step 1:** Prescribing return times: We find a set $A \subset X$ such that the set of first return times to $A$ under $T$ are exactly $n_1, \dots, n_k$.

**Proof Step 1:** Since $\gcd\{n_1, \dots, n_k\} = 1$, let $R$ be a positive integer such that every integer $r \geq R$, can be expressed as nonnegative integer multiples of $n_1, \dots, n_k$.

Following Kakutani's easy proof of Rokhlin's Lemma, let $E$ be a sweep out set (i.e., $X = \cup_{i=0}^{\infty} T^i(E)$), such that the set of first return times back to $E$ are all greater than $R$ (in Step 2 we will require that $E$ has small measure). This is elementary in the case that $T$ is ergodic. See [Hal56] for a proof when $T$ is antiperiodic. Write

$E = \cup_{m \geq R} E_m$, where $E_m = \{x \in E :$ the first return time to $E$ is $m\}$. The column of height $m$ over $E$ is the set $C(E_m) = \cup_{i=1}^{m} T^{i-1} E_m$.

For each $m \geq R$ write $m = qN + r$ where $R \leq r < R + N$ and $N = n_1 n_2 \cdots n_k$. Then we can break up the column of height $m$ over $E$, $(C(E_m))$, into $q$ new subcolumns of height $N$ and one new subcolumn of height $r$ as follows: The first $N$ floors of $C(E_m)$ are labeled $(N, 1), \ldots, (N, N)$, as are the next $N$ floors. After labeling the first $qN$ floors of $C(E_m)$ into $q$ many $N$-columns in this manner, the last $r$ floors are labelled to form an $r$-column (i.e, as $(r, 1), \ldots, (r, r)$). After doing this for each $m \geq R$ and combining all of the new columns of the same height (i.e, grouping all sets with the same label), we end up with one (large) column of height $N$ and at most $N$ "remainder" columns of heights $R, R+1, \ldots R+N-1$.

For each $r$, $R \leq r < R+N$, since we can write $r = r_1 n_1 + \cdots + r_k n_k$ where the $r_i$ are nonnegative integers, we can break up each remainder $r$-column as follows: the first $r_1 n_1$ floors are broken up into $r_1$ columns of height $n_1$; the next block of $r_2 n_2$ floors are grouped into $r_2$ columns of height $n_2$; continuing, the last $r_k n_k$ floors of the $r$-column is broken into $r_k$ columns of height $n_k$.

This partitions the space into a group of columns, one of height $N$ and the others of heights $n_1, \ldots, n_k$. The set $A$ which is the base of these columns, has return times $N$ and $n_1, \ldots, n_k$. Note that the column of height $N$ can be decomposed (labeled) into as many columns of height $n_1, \ldots, n_k$ as we wish, since $N$ is a multiple of $n_i$ for each $i$.

**Step 2:** Controlling the distribution: We note this has been a measure free construction thus far. If we wish to prescribe the distribution of the return times we note that the total measure of the "remainder" columns of height $r, R \leq r < R+N$ is less than $(R+N)\mu(E)$ where $E$ was the base of the original skyscraper. By choosing $E$ small enough so that $(R+N)\mu(E) < \min \{n_i q_i : i = 1, \ldots, k\}$ we can guarantee that after breaking up the remainder columns of height $r$ into columns of heights $n_1, \ldots, n_k$ no column of height $n_i$ has used up more than its "total allowance" of measure $n_i q_i$.

Now partition the $N$-column into $k$ disjoint vertical columns, one for each $i = 1, \ldots, k$, in the following manner: Suppose that the measure of the $n_i$-column used in paving all the "remainder" $r$-columns $(R \leq r < R+N)$ totals up to $p_i$. From the column of height $N$ we take a vertical strip of measure $n_i q_i - p_i$. Then this part of the $N$-column is partitioned into columns of height $n_i$. Choosing disjoint vertical strips from the $N$-column for the different $i$'s, the partition of $X$ into columns of height $n_i, i = 1, \ldots, k$, has the required distribution.

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