New York Journal of Mathematics

New York J. Math. **3** (1997) 1–31.

Norms of Inverses, Spectra, and Pseudospectra of Large Truncated Wiener-Hopf Operators and Toeplitz Matrices

A. Böttcher, S. M. Grudsky, and B. Silbermann

Abstract. This paper is concerned with Wiener-Hopf integral operators on L^p and with Toeplitz operators (or matrices) on l^p. The symbols of the operators are assumed to be continuous matrix functions. It is well known that the invertibility of the operator itself and of its associated operator imply the invertibility of all sufficiently large truncations and the uniform boundedness of the norms of their inverses. Quantitative statements, such as results on the limit of the norms of the inverses, can be proved in the case $p = 2$ by means of C^*-algebra techniques. In this paper we replace C^*-algebra methods by more direct arguments to determine the limit of the norms of the inverses and thus also of the pseudospectra of large truncations in the case of general p.

Contents

Received December 5, 1996.

Mathematics Subject Classification. 47B35.

Key words and phrases. Norms of inverses, Spectral approximation, Pseudospectra, Wiener-Hopf operators, Toeplitz matrices.

The research of the first author was supported by the Alfried Krupp Förderpreis für junge Hochschullehrer of the Krupp Foundation. The research of the second author was supported by the Russian fund of fundamental investigations (Grants N 95-01-01285a and N 96-01-01195a).

ISSN 1076-9803/97

1. Introduction

Given a complex number $c \in \mathbf{C}$ and a function $k \in L^1(\mathbf{R})$, we consider the Wiener-Hopf integral operator W defined by

$$(W\varphi)(x) = c\varphi(x) + \int_0^\infty k(x-t)\varphi(t)\,dt \quad (0 < x < \infty) \tag{1}$$

on $L^p(0,\infty)$ $(1 \le p \le \infty)$. Let Fk be the Fourier transform of k,

$$(Fk)(\xi) = \int_{-\infty}^{\infty} k(x)e^{i\xi x}dx \quad (\xi \in \mathbf{R}).$$

Many properties of the operator (1) can be read off from the function $a = c + Fk$. This function is referred to as the symbol of the operator (1), and we will henceforth denote this operator by $W(a)$. The set $\{c + Fk : c \in \mathbf{C},\ k \in L^1(\mathbf{R})\}$ is denoted by $\mathbf{C} + FL^1(\mathbf{R})$ and called the Wiener algebra.

For $\tau \in (0,\infty)$, the truncated Wiener-Hopf integral operator $W_\tau(a)$ is the compression of $W(a)$ to $L^p(0,\tau)$. Thus, $W_\tau(a)$ acts by the rule

$$\Big(W_\tau(a)\varphi\Big)(x) = c\varphi(x) + \int_0^\tau k(x-t)\varphi(t)\,dt \quad (0 < x < \tau). \tag{2}$$

The norm and the spectrum of an operator A on L^p will be denoted by $\|A\|_p$ and $\sigma_p(A)$, respectively. We write $\|A^{-1}\|_p = \infty$ if A is not invertible on L^p. For $\varepsilon \in (0,\infty)$, the ε-pseudospectrum of an operator A on L^p is defined as the set

$$\sigma_p^\varepsilon(A) = \Big\{\lambda \in \mathbf{C} : \|(A - \lambda I)^{-1}\|_p \ge 1/\varepsilon\Big\}; \tag{3}$$

of course, the points of

$$\sigma_p(A) = \Big\{\lambda \in \mathbf{C} : \|(A - \lambda I)^{-1}\|_p = \infty\Big\}$$

all belong to $\sigma_p^\varepsilon(A)$. In case $W(a)$ or $W_\tau(a)$ is invertible, we denote the inverse by $W^{-1}(a)$ and $W_\tau^{-1}(a)$, respectively.

This paper concerns the limits of $\|W_\tau^{-1}(a)\|_p$, $\sigma_p(W_\tau(a))$, $\sigma_p^\varepsilon(W_\tau(a))$ as τ goes to infinity. We also study $\|A_\tau^{-1}\|$ and $\sigma_p^\varepsilon(A_\tau)$ in case $\{A_\tau\}$ is a more complicated family of operators, say

$$A_\tau = \sum_j \prod_k W_\tau(a_{jk}) \ \text{ or } \ A_\tau = P_\tau \sum_j \prod_k W(a_{jk})P_\tau, \tag{4}$$

where P_τ is given by

$$(P_\tau\varphi)(x) = \begin{cases} \varphi(x) & \text{if} \quad 0 < x < \tau, \\ 0 & \text{if} \quad \tau < x; \end{cases} \tag{5}$$

here and in what follows we freely identify $L^p(0,\tau)$ and the image of P_τ on $L^p(0,\infty)$.

Let $a \in \mathbf{C} + FL^1(\mathbf{R})$. The set $a(\dot{\mathbf{R}}) := \{c\} \cup \{a(\xi) : \xi \in \mathbf{R}\}$ is a closed continuous curve in the complex plane. We give this curve the orientation induced by the change of ξ from $-\infty$ to $+\infty$. For $\lambda \notin a(\dot{\mathbf{R}})$, we denote by $\operatorname{wind}(a,\lambda)$ the winding number of $a(\dot{\mathbf{R}})$ about λ. A classical result says that

$$\sigma_p\Big(W(a)\Big) = a(\dot{\mathbf{R}}) \cup \Big\{\lambda \in \mathbf{C} \setminus a(\dot{\mathbf{R}}) : \operatorname{wind}(a,\lambda) \neq 0\Big\} \tag{6}$$

for $1 \le p \le \infty$ (see [19] and [14]).

Now assume $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $1 \le p \le \infty$. It is well known (see [14, Theorem III.3.1] for $p \in [1,\infty)$ and [24, Theorem 4.58] for $p = \infty$) that

$$\limsup_{\tau\to\infty} \|W_\tau^{-1}(a)\|_p < \infty \tag{7}$$

if and only if $W(a)$ is invertible on $L^p(0,\infty)$. Note that (7) includes the requirement that $W_\tau(a)$ is invertible on $L^p(0,\tau)$ for all sufficiently large $\tau > 0$.

In order to say more about the upper limit in (7), we need the notion of the associated symbol. Given $a = c + Fk \in \mathbf{C} + FL^1(\mathbf{R})$, the associated symbol $\tilde{a} \in \mathbf{C} + FL^1(\mathbf{R})$ is defined by $\tilde{a}(\xi) := a(-\xi)$. Thus, $W(\tilde{a})$ and $W_\tau(\tilde{a})$ may be given by (1) and (2) with $k(x-t)$ replaced by $k(t-x)$. Since $\tilde{a}(\dot{\mathbf{R}})$ differs from $a(\dot{\mathbf{R}})$ only in the orientation, we infer from (6) that $\sigma_p(W(a)) = \sigma_p(W(\tilde{a}))$. In particular, $W(\tilde{a})$ is invertible if and only if $W(a)$ is invertible.

In this paper we prove the following results.

Theorem 1.1. *Let $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $1 \le p < \infty$. If (7) holds then the limit $\lim_{\tau\to\infty} \|W_\tau^{-1}(a)\|_p$ exists and*

$$\lim_{\tau\to\infty} \|W_\tau^{-1}(a)\|_p = \max\Big\{\|W^{-1}(a)\|_p,\ \ \|W^{-1}(\tilde{a})\|_p\Big\}. \tag{8}$$

Theorem 1.2. *There exist $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $p \in (1,\infty)$ such that*

$$\|W^{-1}(\tilde{a})\|_p > \|W^{-1}(a)\|_p. \tag{9}$$

Given a family $\{M_\tau\}_{\tau\in(0,\infty)}$ of subsets M_τ of $\mathbf{C}$, we define the limiting set $\lim_{\tau\to\infty} M_\tau$ as the set of all $\lambda \in \mathbf{C}$ for which there are $\tau_1, \tau_2, \dots$ and $\lambda_1, \lambda_2, \dots$ such that

$$0 < \tau_1 < \tau_2 < \dots, \tau_n \to \infty, \quad \lambda_n \in M_{\tau_n}, \quad \lambda_n \to \lambda.$$

Here is what we will prove about the limiting sets of $\sigma_p(W_\tau(a))$ and $\sigma_p^\varepsilon(W_\tau(a))$.

Theorem 1.3. *Let $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $1 \le p \le \infty$. Then for every $\tau > 0$ the spectrum $\sigma_p(W_\tau(a))$ is independent of p. In particular,*

$$\lim_{\tau\to\infty} \sigma_p\Big(W_\tau(a)\Big) \tag{10}$$

does not depend on p. There exist $a \in \mathbf{C} + FL^1(\mathbf{R})$ such that (10) is not equal to the spectrum of $W(a)$.

Theorem 1.4. *If $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $1 < p < \infty$ then for each $\varepsilon > 0$,*

$$\lim_{\tau\to\infty} \sigma_p^\varepsilon\Big(W_\tau(a)\Big) = \sigma_p^\varepsilon\Big(W(a)\Big) \cup \sigma_p^\varepsilon\Big(W(\tilde{a})\Big). \tag{11}$$

We remark that we will actually prove extensions of Theorems 1.1 and 1.4 which are also applicable to operators as in (4). Furthermore, we will extend Theorems 1.1 and 1.4 to the case of matrix-valued symbols. Finally, we will establish analogues of the above theorems for Toeplitz operators and matrices.

2. Structure of Inverses

In what follows we have to consider the products $W(a)W(b)$ and $W_\tau(a)W_\tau(b)$ and are thus led to algebras generated by Wiener-Hopf operators.

We henceforth exclude the case $p = \infty$ (but see the remark at the end of this section). The precipice between $p \in [1, \infty)$ and $p = \infty$ comes from the fact that the projections P_τ given by (5) converge strongly to the identity operator I if and only if $1 \le p < \infty$.

Given a Banach space X, we denote by $\mathcal{L}(X)$ and $\mathcal{K}(X)$ the bounded and compact (linear) operators on X, respectively. We need a few facts from [13, Section 2] and [8, Chapter 9].

If $a \in L^\infty(\mathbf{R})$, then the operator $\varphi \mapsto F^{-1}aF\varphi$ is bounded on $L^2(\mathbf{R})$. The set of all $a \in L^\infty(\mathbf{R})$ for which there exists a constant $C = C(a,p) < \infty$ such that

$$\|F^{-1}aF\varphi\|_p \le C\|\varphi\|_p \ \text{ for all } \ \varphi \in L^2(\mathbf{R}) \cap L^p(\mathbf{R})$$

is denoted by $M_p(\mathbf{R})$. The set $M_p(\mathbf{R})$ is a Banach algebra with pointwise algebraic operations and the norm

$$\|a\|_{M_p(\mathbf{R})} = \sup\left\{ \frac{\|F^{-1}aF\varphi\|_p}{\|\varphi\|_p} : \varphi \in L^2(\mathbf{R}) \cap L^p(\mathbf{R}), \ \ \varphi \ne 0 \right\}.$$

We have $M_1(\mathbf{R}) = \mathbf{C} + FL^1(\mathbf{R})$ and $M_2(\mathbf{R}) = L^\infty(\mathbf{R})$. If $p \in (1,2) \cup (2,\infty)$ and $1/p + 1/q = 1$, then

$$\mathbf{C} + FL^1(\mathbf{R}) \subset M_p(\mathbf{R}) = M_q(\mathbf{R}) \subset L^\infty(\mathbf{R}),$$

both inclusions being proper. For $a \in M_p(\mathbf{R})$, the Wiener-Hopf integral operator $W(a)$ is defined by

$$W(a)\varphi = \chi_+(F^{-1}aF)\varphi \quad \Big(\varphi \in L^p(0,\infty)\Big),$$

where $(F^{-1}aF)$ is the continuous extension of $F^{-1}aF$ from $L^2 \cap L^p$ to all of L^p and χ_+ denotes the characteristic function of $(0,\infty)$. Of course, if $a \in \mathbf{C} + FL^1(\mathbf{R})$ then $W(a)\varphi$ is given by (1).

One can also associate a Hankel operator $H(a) \in \mathcal{L}(L^p(0,\infty))$ with every $a \in M_p(\mathbf{R})$. We confine ourselves to stating that $H(a)$ can be given by the formula

$$\Big(H(a)f\Big)(x) = \int_0^\infty k(x+t)\varphi(t)\,dt \quad (0 < x < \infty)$$

if $a = c + Fk \in \mathbf{C} + FL^1(\mathbf{R})$.

Let $C_p(\dot{\mathbf{R}})$ denote the closure of $\mathbf{C} + FL^1(\mathbf{R})$ in $M_p(\mathbf{R})$. We have

$$C_1(\dot{\mathbf{R}}) = \mathbf{C} + FL^1(\mathbf{R}), \ \ C_2(\dot{\mathbf{R}}) = C(\dot{\mathbf{R}}),$$

where $C(\dot{\mathbf{R}})$ stands for the continuous functions on $\mathbf{R}$ which have finite and equal limits at $+\infty$ and $-\infty$. If $p \in (1,2) \cup (2,\infty)$ and $1/p + 1/q = 1$, then

$$\mathbf{C} + FL^1(\mathbf{R}) \subset C_p(\dot{\mathbf{R}}) = C_q(\dot{\mathbf{R}}) \subset C(\dot{\mathbf{R}}) \cap M_p(\mathbf{R});$$

again both inclusions being proper. One can show that if $a \in C_p(\dot{\mathbf{R}})$ and $a(\xi) \neq 0$ for all $\xi \in \dot{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, then $a^{-1} \in C_p(\dot{\mathbf{R}})$.

If $a \in M_p(\mathbf{R})$ then $\|W(a)\|_p = |W(a)|_p = \|a\|_{M_p(\mathbf{R})}$, where $|W(a)|_p$ is the essential norm of $W(a)$ on $L^p(0,\infty)$,

$$|W(a)|_p := \inf\Big\{\|W(a)+K\|_p : K \in \mathcal{K}\Big(L^p(0,\infty)\Big)\Big\}.$$

Further, if $a \in C_p(\dot{\mathbf{R}})$ and $K \in \mathcal{K}(L^p(0,\infty))$, then $\sigma_p(W(a))$ is the set (6) and $\sigma_p(W(a)) \subset \sigma_p(W(a)+K)$. In particular, $W(a)$ and a are always invertible in case $W(a)+K$ is invertible.

We denote by $\mathcal{A}_p$ the smallest closed subalgebra of $\mathcal{L}(L^p(0,\infty))$ containing the set $\{W(a) : a \in \mathbf{C} + FL^1(\mathbf{R})\}$. If $a, b \in C_p(\dot{\mathbf{R}})$, then

$$W(a)W(b) = W(ab) - H(a)H(\tilde{b}) \tag{12}$$

and $H(a)$ and $H(\tilde{b})$ (where $\tilde{b}(\xi) = b(-\xi)$) are compact. This and the equality $|W(a)|_p = \|a\|_{M_p(\mathbf{R})}$ imply that every operator in $\mathcal{A}_p$ is of the form $W(a)+K$ with $a \in C_p(\dot{\mathbf{R}})$ and $K \in \mathcal{K}(L^p(0,\infty))$. It is well known that in fact

$$\mathcal{A}_p = \Big\{W(a)+K : a \in C_p(\dot{\mathbf{R}}),\ \ K \in \mathcal{K}\Big(L^p(0,\infty)\Big)\Big\}. \tag{13}$$

Suppose $a \in C_p(\dot{\mathbf{R}})$, $K \in \mathcal{K}(L^p(0,\infty))$, and $W(a)+K$ is invertible. Then $a^{-1} \in C_p(\dot{\mathbf{R}})$, and (12) with $b = a^{-1}$ gives

$$\Big(W(a)+K\Big)W(a^{-1}) = I - H(a)H(\tilde{a}^{-1}) + KW(a^{-1}),$$

which shows that $\Big(W(a)+K\Big)^{-1}$ equals

$$W(a^{-1}) + \Big(W(a)+K\Big)^{-1} H(a)H(\tilde{a}^{-1}) - \Big(W(a)+K\Big)^{-1} KW(a^{-1}).$$

Thus, $(W(a)+K)^{-1}$ is $W(a^{-1})$ plus a compact operator. Note that this observation in particular implies that $\mathcal{A}_p$ is inverse closed in $\mathcal{L}(L^p(0,\infty))$.

The finite section analogue of (12) is Widom's formula

$$W_\tau(a)W_\tau(b) = W_\tau(ab) - P_\tau H(a)H(\tilde{b})P_\tau - R_\tau H(\tilde{a})H(b)R_\tau \tag{14}$$

where P_τ is as in (5) and $R_\tau : L^p(0,\infty) \to L^p(0,\infty)$ is defined by

$$(R_\tau\varphi)(x) = \begin{cases} \varphi(\tau - x) & \text{if} \quad 0 < x < \tau \\ 0 & \text{if} \quad \tau < x \end{cases} \tag{15}$$

(see [34] and [8, 9.43(d)]). Thus, if $a \in C_p(\dot{\mathbf{R}})$, $N \in \mathcal{K}(L^p(0,\infty))$, and $W(a)+N$ is invertible, then (14) with $b = a^{-1}$ yields

$$\begin{aligned}\Big(W_\tau(a) + P_\tau NP_\tau\Big)&W_\tau(a^{-1}) \\ &= P_\tau - P_\tau H(a)H(\tilde{a}^{-1})P_\tau - R_\tau H(a^{-1})H(\tilde{a})R_\tau + P_\tau NP_\tau W_\tau(a^{-1}) \\ &= P_\tau - P_\tau YP_\tau - R_\tau ZR_\tau + P_\tau NP_\tau W_\tau(a^{-1}),\end{aligned}$$

where Y and Z are compact on $L^p(0,\infty)$. Our assumptions on a and N imply that $W_\tau(a) + P_\tau NP_\tau$ is invertible for all sufficiently large τ and that

$$\Big(W_\tau(a) + P_\tau NP_\tau\Big)^{-1} P_\tau \to \Big(W(a)+N\Big)^{-1} \tag{16}$$

strongly on $L^p(0,\infty)$. Hence

$$\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}=W_\tau(a^{-1})+\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}P_\tau YP_\tau$$
$$+\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}R_\tau ZR_\tau+\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}P_\tau NP_\tau W_\tau(a^{-1}).$$

The space $L^p(0,\infty)$ is the dual space of $L^q(0,\infty)$ $(1/p+1/q=1)$ in case $1<p<\infty$ and the dual space of

$$L_0^\infty(0,\infty)=\Big\{\varphi\in L^\infty(0,\infty):\lim_{T\to\infty}\operatorname*{ess\,sup}_{x>T}|\varphi(x)|=0\Big\}$$

in case $p=1$. Since $P_\tau\to I$ strongly on $L^r(0,\infty)$ $(1\le r<\infty)$ and on $L_0^\infty(0,\infty)$ and since Y is compact, we obtain with the help of (16) that

$$\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}P_\tau YP_\tau=P_\tau\Big(W(a)+N\Big)^{-1}YP_\tau+C_\tau^{(1)}$$

where $\|C_\tau^{(1)}\|_p\to 0$ as $\tau\to\infty$. Further, as $R_\tau W_\tau(a)R_\tau=W_\tau(\tilde{a})$ and $R_\tau NR_\tau\to 0$ strongly, we get analogously that

$$\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}R_\tau ZR_\tau=R_\tau\Big(W_\tau(\tilde{a})+R_\tau NR_\tau\Big)^{-1}P_\tau ZR_\tau$$
$$=R_\tau W^{-1}(\tilde{a})ZR_\tau+C_\tau^{(2)}$$

with $\|C_\tau^{(2)}\|_p\to 0$ as $\tau\to\infty$. Finally, in the same way we obtain

$$\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}P_\tau NP_\tau W_\tau(a^{-1})=P_\tau\Big(W_\tau(a)+N\Big)^{-1}NW(a^{-1})P_\tau+C_\tau^{(3)}$$

where $\|C_\tau^{(3)}\|_p\to 0$ as $\tau\to\infty$. In summary, we have shown the following result.

Proposition 2.1. *Let $1\le p<\infty$, $a\in C_p(\dot{\mathbf{R}})$, $N\in\mathcal{K}(L^p(0,\infty))$. If $W(a)+N$ is invertible, then for all sufficiently large τ*

$$\Big(W_\tau(a)+P_\tau NP_\tau\Big)^{-1}=W_\tau(a^{-1})+P_\tau KP_\tau+R_\tau LR_\tau+C_\tau \tag{17}$$

where $K,L\in\mathcal{K}(L^p(0,\infty))$ and $\|C_\tau\|_p\to 0$ as $\tau\to\infty$.

Formula (17) is well known to workers in the field and extraordinarily useful in connection with all questions concerning the behavior of $W_\tau^{-1}(a)$ for large τ. Such formulas were first established and employed in [34] and [30] (also see [18]).

Let $\mathcal{S}_p$ denote the set of all families $\{A_\tau\}=\{A_\tau\}_{\tau\in(0,\infty)}$ of operators $A_\tau\in\mathcal{L}(L^p(0,\tau))$ with $\sup_{\tau>0}\|A_\tau\|_p<\infty$. With obvious algebraic operations and the norm

$$\|\{A_\tau\}\|_p=\sup_{\tau>0}\|A_\tau\|_p,$$

the set $\mathcal{S}_p$ is a Banach algebra. We let $\mathcal{C}_p$ stand for the collection of all elements $\{C_\tau\}\in\mathcal{S}_p$ such that $\|C_\tau\|_p\to 0$ as $\tau\to\infty$. Obviously, $\mathcal{C}_p$ is a closed two-sided ideal of $\mathcal{S}_p$ and if $\{A_\tau\}\in\mathcal{S}_p$, then the norm of the coset $\{A_\tau\}+\mathcal{C}_p$ in the quotient algebra $\mathcal{S}_p/\mathcal{C}_p$ is

$$\|\{A_\tau\}+\mathcal{C}_p\|_p=\limsup_{\tau\to\infty}\|A_\tau\|_p.$$

It is clear that $\{W_\tau(a)\} \in \mathcal{S}_p$ for every $a \in M_p(\mathbf{R})$. We denote by $\mathcal{F}_p$ the smallest closed subalgebra of $\mathcal{S}_p$ containing all families $\{W_\tau(a)\}$ with $a \in \mathbf{C} + FL^1(\mathbf{R})$. If $a, b \in C_p(\dot{\mathbf{R}})$ then all Hankel operators in (14) are compact and hence

$$\{W_\tau(a)\}\{W_\tau(b)\} = \Big\{W_\tau(ab) + P_\tau K P_\tau + R_\tau L R_\tau\Big\}$$

with certain compact operators K and L. This simple observation anticipates the following result.

Proposition 2.2. *If $1 \le p < \infty$ then*

$$\mathcal{F}_p = \bigg\{\Big\{W_\tau(a) + P_\tau K P_\tau + R_\tau L R_\tau + C_\tau\Big\} : \qquad a \in C_p(\dot{\mathbf{R}}), K, L \in \mathcal{K}\big(L^p(0,\infty)\big),\ \{C_\tau\} \in \mathcal{C}_p\bigg\} \tag{18}$$

and $\mathcal{C}_p$ is a closed two-sided ideal of $\mathcal{F}_p$.

The discrete analogue of Proposition 2.2 is proved in [6] and [8, Proposition 7.27]. On first approximating arbitrary families $\{C_\tau\} \in \mathcal{C}_p$ by piecewise constant families, Proposition 2.2 can be proved in the same way as its discrete version.

Remark. For $1 \le p \le \infty$, let $\mathcal{Q}(L^p(0,\infty))$ be the so-called quasi-commutator ideal of $L^p(0,\infty)$, i.e., let $\mathcal{Q}(L^p(0,\infty))$ stand for the smallest closed two-sided ideal of $\mathcal{L}(L^p(0,\infty))$ containing the set

$$\Big\{W(ab) - W(a)W(b) : a, b \in \mathbf{C} + FL^1(\mathbf{R})\Big\}.$$

Proposition 2.1 easily implies that $\mathcal{Q}(L^p(0,\infty)) = \mathcal{K}(L^p(0,\infty))$ if $1 \le p < \infty$. One can show that $\mathcal{Q}(L^\infty(0,\infty))$ is a proper subset of $\mathcal{K}(L^\infty(0,\infty))$ and that, although P_τ does not converge strongly on $L^\infty(0,\infty)$, one has $\|K - P_\tau K\| \to 0$ and $\|K - KP_\tau\| \to 0$ as $\tau \to \infty$ for every $K \in \mathcal{Q}(L^\infty(0,\infty))$ (see [24, Proposition 4.55]). Furthermore, if $K \in \mathcal{Q}(L^\infty(0,\infty))$, then $R_\tau K R_\tau$ goes to zero in the sense of $\mathcal{P}$-convergence (see [24, Section 4.36]). These properties can be used to show that Propositions 2.1 and 2.2 remain true for all $p \in [0,\infty]$ if only $\mathcal{K}(L^p(0,\infty))$ is replaced by $\mathcal{Q}(L^p(0,\infty))$. This in turn yields the validity of some results proved below in the case $p = \infty$ (e.g., of Corollary 3.3 and Theorem 6.2 with $N \in \mathcal{Q}(L^\infty_m(0,\infty))$ and Theorem 7.2 with $N \in \mathcal{Q}(l^\infty_m)$).

3. Norms of Inverses

Let $\mathcal{A}_p$ $(1 \le p < \infty)$ be the algebra (13). We equip the direct sum $\mathcal{A}_p \oplus \mathcal{A}_p$ with the norm

$$\|(A, B)\|_p = \max\big\{\|A\|_p,\ \|B\|_p\big\}.$$

By virtue of Proposition 2.2, we may consider the quotient algebra $\mathcal{F}_p/\mathcal{C}_p$.

Here is the key result of this paper.

Theorem 3.1. *If $1 \le p < \infty$, then the map* $\mathrm{Sym}_p : \mathcal{F}_p/\mathcal{C}_p \to \mathcal{A}_p \oplus \mathcal{A}_p$ *given by*

$$\Big\{W_\tau(a) + P_\tau K P_\tau + R_\tau L R_\tau + C_\tau\Big\} + \mathcal{C}_p \mapsto \Big(W(a) + K,\ W(\tilde{a}) + L\Big)$$

is an isometric Banach algebra homomorphism. Moreover, if $\{A_\tau\} \in \mathcal{F}_p$ then

$$\limsup_{\tau\to\infty} \|A_\tau\|_p = \lim_{\tau\to\infty} \|A_\tau\|_p.$$

Before giving a proof, a few comments are in order. We know from Proposition 2.2 that every family $\{A_\tau\} \in \mathcal{F}_p$ is of the form

$$A_\tau = W_\tau(a) + P_\tau K P_\tau + R_\tau L R_\tau + C_\tau$$

with $K, L \in \mathcal{K}(L^p(0,\infty))$ and $\{C_\tau\} \in \mathcal{C}_p$. Clearly, $W(a) + K$ and $W(\tilde{a}) + L$ are nothing but the strong limits of A_τ and $R_\tau A_\tau R_\tau$, respectively. Thus, we could equivalently also define

$$\mathrm{Sym}_p : \{A_\tau\} + \mathcal{C}_p \mapsto \Big(\mathrm{s}-\lim_{\tau\to\infty} A_\tau,\ \mathrm{s}-\lim_{\tau\to\infty} R_\tau A_\tau R_\tau\Big).$$

This identification of Sym_p shows in particular that Sym_p is a well-defined continuous Banach algebra homomorphism and that

$$\|\mathrm{Sym}_p\big(\{A_\tau\} + \mathcal{C}_p\big)\|_p \le \liminf_{\tau\to\infty} \|A_\tau\|_p. \tag{19}$$

Since the only compact Wiener-Hopf operator is the zero operator, it follows that Sym_p is injective. Thus, Sym_p is a Banach algebra homomorphism and the only thing we must show is that Sym_p is an isometry.

If $p = 2$, then $\mathcal{F}_2/\mathcal{C}_2$ and $\mathcal{A}_2 \oplus \mathcal{A}_2$ are C^*-algebras and Sym_2 is easily seen to be a $*$-homomorphism. Since injective $*$-homomorphisms of C^*-algebras are always isometric, the proof is complete. In the $p \ne 2$ case, the latter conclusion requires hand-work.

Proof. Fix $\varepsilon > 0$. Since $a \in C_p(\dot{\mathbf{R}})$, there is a number $c \in \mathbf{C}$ and a function $k \in L^1(\mathbf{R})$ with finite support, say $\mathrm{supp}\, k \subset (-s, s)$, such that

$$\|\{W_\tau(a - c - Fk)\}\|_p \le \|W(a - c - Fk)\|_p < \varepsilon.$$

Furthermore, if τ_0 is large enough, then $\|P_{\tau_0} K P_{\tau_0} - K\|_p < \varepsilon$, $\|P_{\tau_0} L P_{\tau_0} - L\|_p < \varepsilon$. As $\varepsilon > 0$ may be chosen as small as desired, it follows that we are left with proving that

$$\lim_{\tau\to\infty} \|B_\tau\|_p = \max\Big\{\|W(b) + K\|_p,\ \ \|W(\tilde{b}) + L\|_p\Big\} \tag{20}$$

where

$$B_\tau := W_\tau(b) + P_\tau K P_\tau + R_\tau L R_\tau,$$

$b := c + Fk \in \mathbf{C} + FL^1(\mathbf{R})$, $\mathrm{supp}\, k \subset (-s, s)$, and the compact operators K, L are subject to the condition

$$P_{\tau_0} K P_{\tau_0} = K,\ \ P_{\tau_0} L P_{\tau_0} = L. \tag{21}$$

Here $s > 0$ and $\tau_0 > 0$ are certain fixed numbers.

Pick $l > \max\{\tau_0, s\}$ and put $\tau = 4l$. Let $\varphi \in L^p(0, \tau)$ be any function such that $\|\varphi\|_p = 1$. We claim that there exists a number

$$\beta \in (l + s,\ 3l - s)$$

such that

$$\int_{\beta-s}^{\beta+s} |\varphi(t)|^p dt \leq \left[\frac{l}{s}\right]^{-1} \tag{22}$$

where $[l/s]$ stands for the integral part of l/s. Indeed, since

$$1 = \|\varphi\|_p^p \geq \int_{l}^{3l} |\varphi(t)|^p dt \geq \sum_{d=0}^{d_0-1} \int_{l+2ds}^{l+2(d+1)s} |\varphi(t)|^p dt$$

with $d_0 := [l/s]$, there exists a d_1 such that $0 \leq d_1 \leq d_0 - 1$ and

$$\int_{l+2d_1 s}^{l+2(d_1+1)s} |\varphi(t)|^p dt \leq \left[\frac{l}{s}\right]^{-1},$$

which proves our claim with $\beta := l + (2d_1 + 1)s$.

Given $0 < \tau_1 < \tau_2$, we put $P_{(\tau_1,\tau_2)} := P_{\tau_2} - P_{\tau_1}$. With β as above, we have

$$B_\tau = P_\beta B_\tau P_\beta + P_{(\beta,\tau)} B_\tau P_{(\beta,\tau)} + P_\beta B_\tau P_{(\beta,\tau)} + P_{(\beta,\tau)} B_\tau P_\beta.$$

Since $\beta + \tau_0 < 3l - s + l < 4l = \tau$ and $P_{\tau_0} L P_{\tau_0} = L$, the equality $P_\beta R_\tau L R_\tau P_\beta = 0$ holds. Hence,

$$P_\beta B_\tau P_\beta = P_\beta \Big(W(b) + K \Big) P_\beta,$$

which imples that

$$\|P_\beta B_\tau P_\beta\|_p \leq \|W(b) + K\|_p =: M_1. \tag{23}$$

As $\beta > \tau_0$ and $P_{\tau_0} K P_{\tau_0} = K$, we see that $P_{(\beta,\tau)} P_\tau K P_\tau P_{(\beta,\tau)} = 0$. Therefore,

$$\begin{aligned} P_{(\beta,\tau)} B_\tau P_{(\beta,\tau)} &= P_{(\beta,\tau)} \Big(W_\tau(b) + R_\tau L R_\tau \Big) P_{(\beta,\tau)} \\ &= P_{(\beta,\tau)} R_\tau \Big(W_\tau(\tilde{b}) + L \Big) R_\tau P_{(\beta,\tau)} \\ &= R_\tau P_{\tau-\beta} \Big(W_\tau(\tilde{b}) + L \Big) P_{\tau-\beta} R_\tau \\ &= R_\tau P_{\tau-\beta} \Big(W(\tilde{b}) + L \Big) P_{\tau-\beta} R_\tau, \end{aligned}$$

whence

$$\|P_{(\beta,\tau)} B_\tau P_{(\beta,\tau)}\|_p \leq \|W(\tilde{b}) + L\|_p =: M_2 \tag{24}$$

Again using (21) we get

$$P_{(\beta,\tau)} B_\tau P_\beta + P_\beta B_\tau P_{(\beta,\tau)} = P_{(\beta,\tau)} W_\tau(b) P_\beta + P_\beta W_\tau(b) P_{(\beta,\tau)}.$$

Since $\operatorname{supp} k \subset (-s, s)$, we have

$$\Big(W_\tau(b) P_\beta \varphi \Big)(x) = \int_0^\beta k(x-t)\varphi(t)\, dt = \int_{\beta-s}^{\beta} k(x-t)\varphi(t)\, dt$$

for $x \in (\beta, \tau)$ and

$$\Big(W_\tau(b)P_{(\beta,\tau)}\varphi\Big)(x) = \int_\beta^\tau k(x-t)\varphi(t)\,dt = \int_\beta^{\beta+s} k(x-t)\varphi(t)\,dt$$

for $x \in (0, \beta)$. Consequently,

$$\begin{aligned}
(25)\qquad & \Big\| P_{(\beta,\tau)} B_\tau P_\beta \varphi + P_\beta B_\tau P_{(\beta,\tau)}\varphi \Big\|_p^p \\
&= \int_\beta^\tau \Big| \int_{\beta-s}^{\beta} k(x-t)\varphi(t)\,dt \Big|^p dx + \int_0^\beta \Big| \int_\beta^{\beta+s} k(x-t)\varphi(t)\,dt \Big|^p dx \\
&\le \|k\|_1^p \Big(\int_{\beta-s}^{\beta} |\varphi(t)|^p dt + \int_\beta^{\beta+s} |\varphi(t)|^p dt \Big) \\
&= \|k\|_1^p \int_{\beta-s}^{\beta+s} |\varphi(t)|^p dt \le \|k\|_1^p \left[\frac{l}{s}\right]^{-1},
\end{aligned}$$

the last two estimates resulting from the inequality $\|k * \varphi\|_p \le \|k\|_1 \|\varphi\|_p$ and from (22), respectively. Put

$$\begin{aligned}
& f_1 = P_\beta B_\tau P_\beta \varphi, \quad f_2 = P_{(\beta,\tau)} B_\tau P_{(\beta,\tau)}\varphi, \\
& f_3 = (P_{(\beta,\tau)} B_\tau P_\beta + P_\beta B_\tau P_{(\beta,\tau)})\varphi.
\end{aligned}$$

Then $B_\tau \varphi = f_1 + f_2 + f_3$ and thus,

$$\begin{aligned}
\|B_\tau\varphi\|_p &\le \|f_1 + f_2\|_p + \|f_3\|_p = \Big(\int_0^\tau |f_1(t) + f_2(t)|^p dt \Big)^{1/p} + \|f_3\|_p \\
&= \Big(\int_0^\beta |f_1(t)|^p dt + \int_\beta^\tau |f_2(t)|^p dt \Big)^{1/p} + \|f_3\|_p = \Big(\|f_1\|_p^p + \|f_2\|_p^p \Big)^{1/p} + \|f_3\|_p.
\end{aligned}$$

From (23), (24), (25) we therefore get

$$\begin{aligned}
\|B_\tau\varphi\|_p &\le \Big(M_1^p \|P_\beta\varphi\|_p^p + M_2^p \|P_{(\beta,\tau)}\varphi\|_p^p \Big)^{1/p} + \|f_3\|_p \\
&\le \max\{M_1, M_2\} \Big(\|P_\beta\varphi\|_p^p + \|P_{(\beta,\tau)}\varphi\|_p^p \Big)^{1/p} + \|f_3\|_p \\
&= \max\{M_1, M_2\} \|\varphi\|_p + \|f_3\|_p \\
&\le \max\{M_1, M_2\} + \|k\|_1 [l/s]^{-1}.
\end{aligned}$$

If $\tau \to \infty$ then $[l/s]^{-1} = [\tau/(4s)]^{-1} \to 0$, which gives the estimate

$$\limsup_{\tau\to\infty} \|B_\tau\|_p \le \max\{M_1, M_2\}. \tag{26}$$

Since $\max\{M_1, M_2\} = \|\mathrm{Sym}_p(\{B_\tau\} + \mathcal{C}_p)\|_p$, we finally obtain (20) from (26) and (19). $\square$

Corollary 3.2. *Let* $1 \le p < \infty$ *and* $\{A_\tau\} \in \mathcal{F}_p$. *Put*

$$A = \mathrm{s}-\lim_{\tau\to\infty} A_\tau, \quad B = \mathrm{s}-\lim_{\tau\to\infty} R_\tau A_\tau R_\tau.$$

and suppose $A - \lambda I$ *and* $B - \lambda I$ *are invertible on* $L^p(0,\infty)$. *Then* $A_\tau - \lambda I$ (= $A_\tau - \lambda P_\tau$) *is invertible on* $L^p(0,\tau)$ *for all sufficiently large* τ *and*

$$\lim_{\tau\to\infty} \|(A_\tau - \lambda I)^{-1}\|_p = \max\Big\{\|(A-\lambda I)^{-1}\|_p,\ \|(B-\lambda I)^{-1}\|_p\Big\}.$$

Proof. Since $(A-\lambda I, B-\lambda I) = \mathrm{Sym}_p\Big(\{A_\tau - \lambda I\} + \mathcal{C}_p\Big)$, we deduce from Theorem 3.1 and the inverse closedness of $\mathcal{A}_p$ in $\mathcal{L}(L^p(0,\infty))$ that $\{A_\tau - \lambda I\} + \mathcal{C}_p$ is invertible in $\mathcal{F}_p/\mathcal{C}_p$. Let $\{D_\tau\} + \mathcal{C}_p \in \mathcal{F}_p/\mathcal{C}_p$ be the inverse. Then

$$\mathrm{Sym}_p\Big(\{D_\tau\} + \mathcal{C}_p\Big) = \Big((A-\lambda I)^{-1},\ (B-\lambda I)^{-1}\Big),$$
$$\|\{D_\tau\} + \mathcal{C}_p\|_p = \limsup_{\tau\to\infty} \|(A_\tau - \lambda I)^{-1}\|_p,$$

and therefore the assertion follows from Theorem 3.1. □

The following corollary of Theorem 3.1 clearly contains Theorem 1.1.

Corollary 3.3. *Let* $1 \le p < \infty$, $a \in C_p(\dot{\mathbf{R}})$, $N \in \mathcal{K}(L^p(0,\infty))$. *If* $W(a) + N$ *is invertible on* $L^p(0,\infty)$ *then the limit*

$$\lim_{\tau\to\infty} \|\big(W_\tau(a) + P_\tau N P_\tau\big)^{-1}\|_p \tag{27}$$

exists and is equal to

$$\max\Big\{\|\big(W(a)+N\big)^{-1}\|_p,\ \|W^{-1}(\tilde{a})\|_p\Big\}. \tag{28}$$

Proof. Theorem 3.1 and Proposition 2.1 show that the limit (27) exists and equals

$$\max\Big\{\|W(a^{-1}) + K\|_p,\ \|W(\tilde{a}^{-1}) + L\|_p\Big\}.$$

Passing in (17) to the strong limit $\tau \to \infty$ we get $\Big(W(a) + N\Big)^{-1} = W(a^{-1}) + K$, and multiplying (17) from both sides by R_τ and then passing to the strong limit $\tau \to \infty$ we see that $W^{-1}(\tilde{a}) = W(\tilde{a}^{-1}) + L$. □

Corollary 3.4. *Let* $1 \le p < \infty$ *and let* a_{jk} *be a finite collection of functions in* $C_p(\dot{\mathbf{R}})$. *Put*

$$A = \sum_j \prod_k W(a_{jk}), \quad B_1 = \sum_j \prod_k W(\tilde{a}_{jk}), \quad B_2 = W\Big(\sum_j \prod_k \tilde{a}_{jk}\Big).$$

and suppose A *is invertible on* $L^p(0,\infty)$

(a) *If* B_1 *is invertible on* $L^p(0,\infty)$ *then*

$$\lim_{\tau\to\infty} \Big\|\Big(\sum_j \prod_k W_\tau(a_{jk})\Big)^{-1}\Big\|_p = \max\Big\{\|A^{-1}\|_p,\ \|B_1^{-1}\|_p\Big\}, \tag{29}$$

while if B_1 is not invertible on $L^p(0,\infty)$ then

(30)
$$\lim_{\tau\to\infty}\Big\|\Big(\sum_j\prod_k W_\tau(a_{jk})\Big)^{-1}\Big\|_p=\infty.$$

(b) *The operator B_2 is invertible and*

(31)
$$\lim_{\tau\to\infty}\Big\|\Big(P_\tau\sum_j\prod_k W(a_{jk})P_\tau\Big)^{-1}\Big\|=\max\Big\{\|A^{-1}\|_p,\,\|B_2^{-1}\|_p\Big\}.$$

Proof. Let $h=\sum_j\prod_k a_{jk}$.

(a) We know from Section 2 that $A=W(h)+M$ and $B_1=W(\tilde{h})+N$ with compact operators M and N. Obviously,

(32)
$$A_\tau:=\sum_j\prod_k W_\tau(a_{jk})\to A \;\text{ strongly}$$

and

(33)
$$R_\tau A_\tau R_\tau=\sum_j\prod_k W_\tau(\tilde{a}_{jk})\to B_1 \;\text{ strongly}.$$

Suppose the limit in (30) does not exist or is finite. Then there are $\tau_k\to\infty$ such that $\|(R_{\tau_k}A_{\tau_k}R_{\tau_k})^{-1}\|_p=\|A_{\tau_k}^{-1}\|_p\le m<\infty$. This implies that $\|P_{\tau_k}\varphi\|_p\le m\|R_{\tau_k}A_{\tau_k}R_{\tau_k}\varphi\|_p$ and thus $\|\varphi\|_p\le m\|B_1\varphi\|_p$ for all $\varphi\in L^p(0,\infty)$. Consequently, B_1 is injective. Since $A=W(h)+M$ is assumed to be invertible, the operator $B_1=W(\tilde{h})+N$ must be Fredholm of index zero (see [14]). This in conjunction with the injectivity of B_1 shows that B_1 is actually invertible. Thus, if B_1 is not invertible, then (30) holds.

Now suppose B_1 is invertible. From (32) and (33) we infer that $\mathrm{Sym}_p\Big(\{A_\tau\}+\mathcal{C}_p\Big)$ equals (A,B_1), so that (29) is immediate from Theorem 3.1.

(b) If $A=W(h)+M$ is invertible, then so also is $B_2=W(\tilde{h})$ and (31) is nothing but the conclusion of Corollary 3.2. □

For $p=2$, Theorem 3.1 was established in [31], and Corollaries 3.2 to 3.4 appeared explicitly in [4] for the first time. It should be noted that [4] and [31] deal with Wiener-Hopf operators generated by piecewise continuous symbols, in which case the analogues of the algebras $\mathcal{A}_p$ and $\mathcal{F}_p/\mathcal{C}_p$ do not admit such simple descriptions as in (13) and (18). The proofs given in [4] and [31] make heavy use of the C^*-algebra techniques which have their origin in [6] (also see [5], [8], [17], [28], [32]). We remark that the aforementioned works raised some psychological barrier, in view of which the validity of such $p\ne 2$ results as Theorem 3.1 is fairly surprising. The psychological barrier was broken only in [16], where (the discrete analogue of) the $N=0$ version of Corollary 3.3 was established for $p=1$ by different methods. In particular, in view of the proofs of [16] and the proof of Theorem 3.1 given here, one of the authors withdraws his too enthusiastic statement that "formulas like

$$\limsup_{\tau\to\infty}\|W_\tau{}^{-1}(a)\|_2=\|W^{-1}(a)\|_2$$

cannot be proved by bare hands", which is on p. 274 of [4]. Finally, it should be mentioned that the l^p version of Theorem 3.1 was de facto predicted already in [27, Remark 3 on p. 303] and is in very disguised form also contained in the results of [17, pp. 186–205].

We remark that Corollary 3.3 can be stated in slightly different terms. Define the sesquilinear operator R on $L^p(0,\infty)$ by $(R\varphi)(x) = \overline{\varphi(x)}$. Obviously,

$$R^2 = I \text{ and } W(\tilde{a}) = RW(\overline{a})R$$

where $\overline{a}(\xi) = \overline{a(\xi)}$. Notice that if $a = c + Fk \in \mathbf{C} + FL^1(\mathbf{R})$, then $W(\overline{a})$ and $W_\tau(\overline{a})$ are given by (1) and (2) with $\overline{k(t-x)}$ in place of $k(x-t)$. Since R is an isometry, we have

$$\|W^{-1}(\tilde{a})\|_p = \|W^{-1}(\overline{a})\|_p \tag{34}$$

for $1 \le p \le \infty$. On identifying the dual space of L^p $(1 < p < \infty)$ with L^q $(1/p + 1/q = 1)$, we may think of $W(\overline{a})$ as the adjoint operator of $W(a)$. Thus,

$$\|W^{-1}(\overline{a})\|_p = \|W^{-1}(a)\|_q \tag{35}$$

for $1 < p < \infty$, and in Theorem 1.1 and Corollary 3.3 we can replace $\|W^{-1}(\tilde{a})\|_p$ by $\|W^{-1}(a)\|_q$ in case $1 < p < \infty$. If $p = 2$, then Corollary 3.3 implies that

$$\lim_{\tau\to\infty} \|W^{-1}(a)\|_2 = \|W^{-1}(a)\|_2. \tag{36}$$

Theorem 1.2 shows that in the above results the maximum cannot be removed.

We remark that the previous results extend to certain spaces with weights. Given a real number μ and a measurable function f on $(0,\infty)$, we define

$$(\Lambda_\mu f)(x) := (1+|x|)^\mu f(x).$$

Let $L^{p,\mu}(0,\infty)$ stand for the measurable functions f on $(0,\infty)$ satisfying

$$\|f\|_{p,\mu} := \|\Lambda_\mu f\|_p < \infty,$$

and let $L^{1,\mu}(\mathbf{R})$ be the space of all measurable functions on $\mathbf{R}$ for which $\Lambda_\mu f$ is in $L^1(\mathbf{R})$. If $a \in \mathbf{C} + FL^{1,|\mu|}(\mathbf{R})$, then $W(a)$ and $W_\tau(a)$ are bounded on $L^{p,\mu}(0,\infty)$ and $L^{p,\mu}(0,\tau)$ for all $p \in [1,\infty)$ and all $\mu \in \mathbf{R}$. The algebras $\mathcal{A}_{p,\mu}$ and $\mathcal{F}_{p,\mu}$ are defined in the natural manner. Taking into account [24, Remark 4.51, 2°] one can construct an isometric Banach algebra isomorphism of $\mathcal{F}_{p,\mu}/\mathcal{C}_{p,\mu}$ onto the direct sum $\mathcal{A}_{p,\mu} \oplus \mathcal{A}_p$. It follows in particular that if $a \in \mathbf{C} + FL^{1,|\mu|}(\mathbf{R})$, $a(\xi) \neq 0$ for all $\xi \in \dot{\mathbf{R}}$, and wind $(a,0) = 0$, then

$$\limsup_{\tau\to\infty} \|W_\tau^{-1}(a)\|_{p,\mu} = \lim_{\tau\to\infty} \|W_\tau^{-1}(a)\|_{p,\mu} = \max\Big\{\|W^{-1}(a)\|_{p,\mu},\ \|W^{-1}(\tilde{a})\|_p\Big\}.$$

Proof of Theorem 1.2. For $\xi \in \mathbf{R}$, put

$$a_-(\xi) = 1 + \frac{i}{2(\xi - i)}, \quad a_+(\xi) = 1 + \frac{2i}{\xi + i} - \frac{6}{(\xi+i)^2}$$

and consider $a = a_-^{-1}a_+^{-1}$. Since $a_\pm \in \mathbf{C} + FL^1(\mathbf{R})$ and these functions have no zeros on $\mathbf{R}$, the symbol a belongs to $\mathbf{C} + FL^1(\mathbf{R})$ by Wiener's theorem. The functions a_- and a_+ have no zeros in the lower and upper complex half-planes, respectively, which implies (see, e.g., [14]) that $W(a)$ is invertible on $L^p(0,\infty)$ for $1 \le p \le \infty$ and that $W^{-1}(a) = W(a_+)W(a_-)$. Consequently,

$$\Big(W^{-1}(a)\varphi\Big)(x) = \varphi(x) + \int_0^\infty \gamma(x,t)\varphi(t)\,dt \quad (0 < x < \infty) \tag{37}$$

where

$$\gamma(x,t) = \gamma_+(x-t) + \gamma_-(x-t) + \int_0^{\min\{t,x\}} \gamma_+(x-r)\gamma_-(r-t)\,dr$$

and $\gamma_\pm$ are the kernels defined by $a_\pm = 1 + F\gamma_\pm$. Clearly,

$$\gamma_+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2i}{\xi+i} - \frac{6}{(\xi+i)^2}\right)e^{-i\xi x}d\xi = \begin{cases} 2e^{-x} + 6xe^{-x} & \text{if} \quad x>0, \\ 0 & \text{if} \quad x<0, \end{cases}$$

$$\gamma_-(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i}{2(\xi-i)}e^{-i\xi x}d\xi = \begin{cases} -\frac{1}{2}e^{x} & \text{if} \quad x<0, \\ 0 & \text{if} \quad x>0, \end{cases}$$

whence

$$\gamma(x,t) = \frac{1}{4}\begin{cases} (3+18(x-t))e^{-(x-t)} + (5+6x)e^{-(x+t)} & \text{if} \quad x>t, \\ -7e^{x-t} + (5+6x)e^{-(x+t)} & \text{if} \quad x<t. \end{cases}$$

From the representation (37) we infer that

$$\|W^{-1}(a)\|_1 = 1 + \sup_{t>0}\int_0^\infty |\gamma(x,t)|\,dx, \quad \|W^{-1}(a)\|_\infty = 1 + \sup_{x>0}\int_0^\infty |\gamma(x,t)|\,dt.$$

It is obvious that $\gamma(x,t) > 0$ for $x > t$, and taking into account that $f(x) = (5+6x)e^{-2x}$ is monotonically decreasing on $(0,\infty)$ it is easily seen that $\gamma(x,t) < 0$ for $x < t$. Thus, a straightforward computation gives

$$S_1(t) := \int_0^\infty |\gamma(x,t)|\,dt = 7 - \frac{9}{2}e^{-t} + \left(\frac{11}{2} + 3t\right)e^{-2t},$$

$$S_\infty(x) := \int_0^\infty |\gamma(x,t)|\,dt = 7 - (4+3x)e^{-x} - \left(\frac{5}{2} + 3x\right)e^{-2x}.$$

We have $S_1(0) = 8$, $S_1(\infty) = 7$, and $S_1'(t) = e^{-2t}\Big((9/2)e^t - (8+6t)\Big)$. Let $t_0 > 0$ be the unique solution of the equation $(9/2)e^t = 8 + 6t$. Then $S_1(t)$ is monotonically decreasing on $(0,t_0)$ and monotonically increasing on (t_0,∞), which implies that

$$\|W^{-1}(a)\|_1 = 1 + \sup_{t>0} S_1(t) = 1 + S_1(0) = 9. \tag{38}$$

Since $S_\infty'(x) = (1+3x)e^{-x} + (2+6x)e^{-2x} > 0$ for all $x \in (0,\infty)$, it follows that

$$\|W^{-1}(a)\|_\infty = 1 + \sup_{x>0} S_\infty(x) = 1 + S_\infty(\infty) = 8. \tag{39}$$

We now show that if $p \in (1,\infty)$ is sufficiently large and $1/p + 1/q = 1$, then

$$\|W^{-1}(a)\|_p < \|W^{-1}(a)\|_q. \tag{40}$$

This together with (34) and (35) gives the desired inequality (9).

Abbreviate $W^{-1}(a)$ to A. By (38), there is a function $f \in L^\infty(\mathbf{R})$ with finite support, say $\operatorname{supp} f \subset (0,T)$, such that $\|Af\|_1/\|f\|_1 > 9-\varepsilon$. Since

$$\|f\|_{1+\delta}^{1+\delta} = \int_0^T |f(x)|^{1+\delta} dx$$

depends continuously on $\delta \in [0,\infty)$, we see that $\|f\|_{1+\delta}^{1+\delta} \to \|f\|_1$ as $\delta \to 0$, whence $\|f\|_{1+\delta} \to \|f\|_1$ as $\delta \to 0$. Further, we know that

$$(Af)(x) = f(x) + \int_0^T \gamma(x,t) f(t)\, dt.$$

If $x > T$, then $\int_0^T \gamma(x,t)f(t)\,dt$ equals

$$\int_0^T \Big(3 + 18(x-t)\Big) e^{-x} e^t f(t)\, dt + \int_0^T (5+6x)e^{-x}e^t f(t)\, dt,$$

which is of the form $\alpha e^{-x} + \beta x e^{-x}$ with $\alpha, \beta \in \mathbf{C}$. Consequently,

$$(Af)(x) = g(x) + (\alpha + \beta x)e^{-x}$$

where $g \in L^\infty(0,\infty)$ and $\operatorname{supp} g \subset (0,T)$. This easily implies that $\|Af\|_{1+\delta}^{1+\delta} \to \|Af\|_1$ and thus $\|Af\|_{1+\delta} \to \|Af\|_1$ as $\delta \to 0$.

In summary, we have shown that $\|Af\|_{1+\delta}/\|f\|_{1+\delta} > 9 - 2\varepsilon$ and thus $\|A\|_{1+\delta} > 9 - 2\varepsilon$ whenever $\delta > 0$ is sufficiently small. From the Riesz-Thorin interpolation theorem and from (38), (39) we obtain that

$$\|A\|_p \le \|A\|_1^{1/p} \|A\|_\infty^{1-1/p} = 9^{1/p} 8^{1-1/p} < 8 + \varepsilon$$

if only p is sufficiently large. But if p is large enough, then $q = 1+\delta$ with δ as small as desired. As $8 + \varepsilon < 9 - 2\varepsilon$ for $\varepsilon < 1/3$, we arrive at (40). □

4. Spectra

The comparision of the spectra of $W(a)$ and $W_\tau(a)$ as well as the discrete analogue of this problem, the comparision of the spectra of infinite Toeplitz matrices and their large finite sections, has been the subject of extensive investigations for many decades and is nevertheless a field with still a lot of mysteries. Clearly, the Szegö-Widom theorem and its continuous analogue, the Achiezer-Kac-Hirschman formula, pertain to this topic (see the books [7], [8], [15], [35]). The asymptotic spectral behavior of truncated Toeplitz and Wiener-Hopf operators is explicitly discussed in the papers [1], [2], [4], [5], [9], [12], [23], [27], [29], [36], [33]; this list is incomplete. We here confine ourselves to pointing out only a couple of phenomena.

The equality (6) holds for every $a \in C_p(\dot{\mathbf{R}})$ and shows that $\sigma_p(W(a))$ is independent of p (note that this is no longer true for piecewise continuous symbols). The following result confirms the same effect for truncated Wiener-Hopf operators.

Theorem 4.1. *If $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $\tau > 0$, then $\sigma_p(W_\tau(a))$ does not depend on $p \in [1,\infty]$.*

Proof. Define the shift operators V_τ and $V_\tau^{(-1)}$ on $L^p(0,\infty)$ by

$$(V_\tau\varphi)(x) = \begin{cases} \varphi(x-\tau) & \text{if} \quad \tau < x \\ 0 & \text{if} \quad 0 < x < \tau \end{cases}, \quad (V_\tau^{(-1)}\varphi)(x) = \varphi(x+\tau).$$

We think of the elements of the direct sum $L^p(0,\infty) \oplus L^p(0,\infty) =: L_2^p(0,\infty)$ as column vectors and consider the operator B given on $L_2^p(0,\infty)$ by the operator matrix

$$B = \begin{pmatrix} V_\tau & W(a) \\ 0 & V_\tau^{(-1)} \end{pmatrix}.$$

It is easily seen that $W_\tau(a)$ is invertible on $L^p(0,\tau)$ if and only if B is invertible on $L_2^p(0,\infty)$.

Let $a = c + Fk$ with $c \in \mathbf{C}$ and $k \in L^1(\mathbf{R})$. The operator B may be identified with the block Wiener-Hopf operator $W(b)$ whose symbol is

$$b(\xi) = \begin{pmatrix} e^{i\tau\xi} & a(\xi) \\ 0 & e^{-i\tau\xi} \end{pmatrix}$$

(see [14]). Letting

$$m_-(\xi) := \begin{pmatrix} -c & 0 \\ e^{i\tau\xi} & 1 \end{pmatrix}, \quad m_+(\xi) := \begin{pmatrix} 1 & 0 \\ e^{-i\tau\xi} & -c \end{pmatrix},$$

we can write $W(m_-)W(b)W(m_+) = W(d)$ where

$$\begin{aligned} d(\xi) &= m_-(\xi)b(\xi)m_+(\xi) \\ &= \begin{pmatrix} e^{i\tau\xi}(Fk)(\xi) & c + (Fk)(\xi) \\ -c + (Fk)(\xi) & e^{-i\tau\xi}(Fk)(\xi) \end{pmatrix} \\ &= \begin{pmatrix} (Fk_\tau)(\xi) & c + (Fk)(\xi) \\ -c + (Fk)(\xi) & (Fk_{-\tau})(\xi) \end{pmatrix} \end{aligned}$$

with $k_\tau(x) = k(x-\tau)$, $k_{-\tau}(x) = k(x+\tau)$. The operators $W(m_\pm)$ are invertible, the inverses being $W(m_\mp^{-1})$. Thus, $B = W(b)$ is invertible if and only if $W(d)$ is invertible.

The operator $W(d)$ is a block Wiener-Hopf operator and the symbols of the entries of $W(d)$ belong to $\mathbf{C} + FL^1(\mathbf{R})$. We have $\det d(\xi) = c^2$, and hence $W(d)$ is invertible if and only if $c \neq 0$ and the partial indices of d are both zero (again see [14]). As the partial indices of a nonsingular matrix function with entries in the Wiener algebra do not depend on p, we arrive at the conclusion that the invertibility of our original operator $W_\tau(a)$ on $L^p(0,\infty)$ is independent of p.

Since $W_\tau(a) - \lambda I = W_\tau(a-\lambda)$, we finally see that the spectrum $\sigma_p(W_\tau(a))$ is the same for all $p \in [1,\infty]$. □

Thus, when considering the limiting set

$$\Lambda(a) := \lim_{\tau\to\infty} \sigma_p\Big(W_\tau(a)\Big)$$

for $a \in \mathbf{C} + FL^1(\mathbf{R})$ we can restrict ourselves to the case $p = 2$.

For Toeplitz matrices with rational symbols the discrete analogue of $\Lambda(a)$ was completely identified by Schmidt and Spitzer [29] and Day [11], [12]. The Wiener-Hopf analogue of this result was established in [9] with the help of formulas for

Wiener-Hopf determinants contained in [3]. We only remark that if $a \in \mathbf{C}+FL^1(\mathbf{R})$ is a rational function, then $\Lambda(a)$ is a nonempty bounded set which is comprised of a finite union of closed analytic arcs and which is in no obvious way related to the spectrum of $W(a)$. For example, if $a(\xi) = (3+i\xi)/(1+\xi^2)$, then $\sigma_p(W(a))$ is a set bounded by an ellipse while $\Lambda(a)$ is the union of the circle $\{\lambda \in \mathbf{C} : |\lambda - 1/12| = 1/12\}$ and the interval $[3/2-\sqrt{2}, 3/2+\sqrt{2}]$ (see [1] and [9]). Note that Theorem 4.1 and this example completes the proof of Theorem 1.3.

The situation is more delicate for non-rational symbols. Curiously, in this case the behavior of $\sigma_p(W_\tau(a))$ is often more "canonical" than one might expect. Let, for example, $a(\xi) = -\operatorname{sign}\xi$. This is a piecewise continuous function with two jumps, one at the origin and one at infinity. The Wiener-Hopf operator associated with this function is the Cauchy singular integral operator

$$\Big(W(a)\varphi\Big)(x) = \frac{1}{\pi i}\int_0^\infty \frac{\varphi(t)}{t-x}\,dt \quad (0 < x < \infty).$$

Of course, $W_\tau(a)$ is given by the same formula with ∞ replaced by τ. It is well known (see, e.g., [13]) that if $1 < p < \infty$, then

$$\sigma_p\Big(W_\tau(a)\Big) = \sigma_p\Big(W(a)\Big) = \mathcal{O}_p(-1,1) \text{ for all } \tau > 0,$$

where $\mathcal{O}_p(-1,1)$ is the set of all $\lambda \in \mathbf{C}$ at which the line segment $[-1,1]$ is seen at an angle θ satisfying

$$\max\{2\pi/p,\, 2\pi/q\} \le \theta \le \pi \quad (1/p+1/q=1).$$

Thus, we are in the best of all possible cases. In a sense, it is continuous and non-rational (or non-analytic) symbols which cause real problems. We refer to [36] and [2] for a discussion of this phenomenon.

5. Pseudospectra

Things are dramatically simpler when passing from spectra to pseudospectra. Henry Landau [20], [21], [22] was probably the first to study pseudospectra of truncated Toeplitz and Wiener-Hopf operators. For $p = 2$, the discrete analogue of Theorem 1.4 was established by Reichel and Trefethen [26], although their proof contained a gap. A completely different proof (based on C^*-algebra techniques) was given in [4]. Pseudospectra of Wiener-Hopf integral operators (with Volterra kernels) were also considered in [25]. We remark that in [4] the formula (11) and its discrete analogue are proved for $p = 2$ under the sole assumption that a is a locally normal function, which is, for example, the case if a is piecewise continuous.

We will derive formula (11) from Corollary 3.2. In order to do this, we need the following result, which says that the norm of the resolvent of an operator on L^p cannot be locally constant. For $p = 2$, we learned this result from Andrzej Daniluk (private communication, see [10]); his proof is published in [4]. We here give a proof for general p.

Theorem 5.1. *Let $(X, d\mu)$ be a space with a measure and let $1 < p < \infty$. Suppose A is a bounded linear operator on $L^p(X, d\mu)$ and $A - \lambda I$ is invertible for all λ in some* *open subset U of $\mathbf{C}$. If $\|(A-\lambda I)^{-1}\|_p \le M$ for all $\lambda \in U$, then $\|(A-\lambda I)^{-1}\|_p < M$ for all $\lambda \in U$.*

Remark. It should be noted that such a result is not true for general analytic operator-valued functions. To see this, equip $\mathbf{C}^2$ with the norm $\left\| \begin{pmatrix} z \\ \omega \end{pmatrix} \right\|_p = \sqrt[p]{|z|^p + |\omega|^p}$ and consider the function

$$A : \mathbf{C} \to \mathcal{L}(\mathbf{C}^2), \quad \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $\|A(\lambda)\|_p = \max\{|\lambda|, 1\}$ and thus $\|A(\lambda)\|_p = 1$ for all λ in the unit disk.

Proof. We may without loss of generality assume that $p \ge 2$; otherwise we can pass to the adjoint operator.

A little thought reveals that what we must show is the following: if U is an open subset of $\mathbf{C}$ containing the origin and $\|(A-\lambda I)^{-1}\|_p \le M$ for all $\lambda \in U$, then $\|A^{-1}\|_p < M$. Assume the contrary, i.e., let

$$\|A^{-1}\|_p = M. \tag{41}$$

We have

$$(A-\lambda I)^{-1} = \sum_{j=0}^{\infty} \lambda^j A^{-(j+1)}$$

for all $\lambda = re^{i\varphi}$ in some disk $|\lambda| = r \le r_0$. Hence, for every $f \in L^p(X, d\mu)$,

$$\begin{aligned} \|(A-\lambda I)^{-1} f\|_p^p &= \int_X \left| \sum_{j=0}^{\infty} \lambda^j (A^{-(j+1)} f)(x) \right|^p d\mu(x) \\ &= \int_X \left| \sum_{j=0}^{\infty} r^j e^{ij\varphi} (A^{-(j+1)} f)(x) \right|^{2p/2} d\mu(x) \\ &= \int_X \left| C(r,x) + \sum_{l=1}^{\infty} B_l(r,x,\varphi) \right|^{p/2} d\mu(x) \end{aligned} \tag{42}$$

with

$$C(r,x) = \sum_{j=0}^{\infty} r^{2j} \left| (A^{-(j+1)} f)(x) \right|^2,$$

$$B_l(r,x,\varphi) = 2 \sum_{k=0}^{\infty} r^{l+2k} \mathrm{Re} \left(e^{il\varphi} (A^{-(l+k+1)} f)(x) \overline{(A^{-(k+1)} f)(x)} \right).$$

For $n = 0, 1, 2, \dots$, put

$$I_n(r,\varphi,f) = \int_X \left| C(r,x) + \sum_{l=1}^{\infty} B_{2^n l}(x,r,\varphi) \right|^{p/2} d\mu(x). \tag{43}$$

We now apply the inequality

$$\frac{|a+b|^{p/2} + |a-b|^{p/2}}{2} \ge |a|^{p/2} \tag{44}$$

to

$$a = C(r,x) + \sum_{l=0}^{\infty} B_{2l}(r,x,\varphi), \quad b = \sum_{l=1}^{\infty} B_{2l-1}(r,x,\varphi)$$

and integrate the result over X. Taking into account that

$$\int_X |a-b|^{p/2} d\mu(x) = I_0(r,\varphi+\pi,f),$$

we get

$$\frac{I_0(r,\varphi,f) + I_0(r,\varphi+\pi,f)}{2} \geq I_1(r,\varphi,f). \tag{45}$$

Letting

$$a = C(r,x) + \sum_{l=1}^{\infty} B_{4l}(r,x,\varphi), \quad b = \sum_{l=1}^{\infty} B_{4l-2}(r,x,\varphi)$$

in (44), we analogously obtain after integration that

$$\frac{I_1(r,\varphi,f) + I_1(r,\varphi+\pi/2,f)}{2} \geq I_2(r,\varphi,f). \tag{46}$$

Combining (45) and (46) we arrive at the inequality

$$\begin{aligned} &\frac{I_0(r,\varphi,f) + I_0(r,\varphi+\pi/2,f) + I_0(r,\varphi+\pi,f) + I_0(r,\varphi+3\pi/2,f)}{4} \\ &\qquad \geq \frac{I_1(r,\varphi,f) + I_1(r,\varphi+\pi/2,f)}{2} \geq I_2(r,\varphi,f). \end{aligned} \tag{47}$$

In the same way as above we see that

$$\frac{I_2(r,\varphi,f) + I_2(r,\varphi+\pi/4,f)}{2} \geq I_3(r,\varphi,f),$$

which together with (47) gives

$$\frac{1}{8}\sum_{m=0}^{7} I_0\Big(r,\varphi+\frac{m\pi}{4},f\Big) \geq I_3(r,\varphi,f).$$

Obviously, continuing this process we obtain

$$\frac{1}{2^n}\sum_{m=0}^{2^n-1} I_0\left(r,\varphi+\frac{m\pi}{2^{n-1}},f\right) \geq I_n(r,\varphi,f) \tag{48}$$

for every $n \geq 0$.

Now put $\varphi = 0$ in (48). From (43) we infer that the right-hand side of (48) converges to

$$\int_X |C(r,x)|^{p/2} d\mu(x)$$

as $n \to \infty$. The left-hand side of (48) is an integral sum and hence passage to the limit $n \to \infty$ gives

$$\lim_{n\to\infty} \frac{1}{2\pi}\frac{\pi}{2^{n-1}}\sum_{m=0}^{2^n-1} I_0\left(r,\frac{m\pi}{2^{n-1}},f\right) = \frac{1}{2\pi}\int_0^{2\pi} I_0(r,\varphi,f)\,d\varphi.$$

As $I_0(r,\varphi,f) = \|(A - re^{i\varphi}I)^{-1}f\|_p^p$ by (42), it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \|(A - re^{i\varphi}I)^{-1}f\|_p^p d\varphi \ge \int_X |C(r,x)|^{p/2} d\mu(x). \tag{49}$$

Assume $\|f\|_p = 1$. Then our hypothesis $\|(A-\lambda I)^{-1}\|_p \le M$ and (49) imply that

$$\begin{aligned} M^p &\ge \frac{1}{2\pi}\int_0^{2\pi} \|(A - re^{i\varphi}I)^{-1}f\|_p^p d\varphi \ge \int_X |C(r,x)|^{p/2} d\mu(x) \\ &\ge \int_X \Big(|(A^{-1}f)(x)|^2 + r^2|(A^{-2}f)(x)|^2\Big)^{p/2} d\mu(x). \end{aligned}$$

Since $(|a|+|b|)^{p/2} \ge |a|^{p/2} + |b|^{p/2}$, we get

$$\begin{aligned} M^p &\ge \int_X |(A^{-1}f)(x)|^p d\mu(x) + r^p \int_X |(A^{-2}f)(x)|^p d\mu(x) \\ &= \|A^{-1}f\|_p^p + r^p\|A^{-2}f\|_p^p. \end{aligned} \tag{50}$$

Let $\varepsilon > 0$ be an arbitrary number. By virtue of (41), we can find an $f \in L^p(X, d\mu)$ such that $\|f\|_p = 1$ and $\|A^{-1}f\|_p^p \ge M^p - \varepsilon$. Since

$$\|A^{-2}f\|_p \ge \|A^2\|_p^{-1}\|f\|_p = \|A^2\|_p^{-1},$$

we finally obtain from (50) that

$$M^p \ge M^p - \varepsilon + r^p\|A^2\|_p^{-p} \text{ for all } r \in (0, r_0],$$

which is impossible if ε is sufficiently small. This contradiction completes the proof. □

Theorem 5.2. *Let $1 < p < \infty$ and $A_\tau = W_\tau(a) + P_\tau K P_\tau + R_\tau L R_\tau + C_\tau$ with a in $C_p(\dot{\mathbf{R}})$, $K, L \in \mathcal{K}(L^p(0,\infty))$, and $\|C_\tau\|_p \to 0$ as $\tau \to \infty$. Then for each $\varepsilon > 0$,*

$$\lim_{\tau\to\infty} \sigma_p^\varepsilon(A_\tau) = \sigma_p^\varepsilon\Big(W(a) + K\Big) \cup \sigma_p^\varepsilon\Big(W(\tilde{a}) + L\Big). \tag{51}$$

Proof. Put $A = W(a) + K$ and $B = W(\tilde{a}) + L$. We first show the inclusion

$$\sigma_p^\varepsilon(A) \subset \lim_{\tau\to\infty} \sigma_p^\varepsilon(A_\tau). \tag{52}$$

Pick $\lambda \in \sigma_p(A)$. We claim that

$$\limsup_{\tau\to\infty} \|(A_\tau - \lambda I)^{-1}\|_p = \infty. \tag{53}$$

Assume the contrary, i.e., let $\|(A_\tau - \lambda I)^{-1}\|_p \le m < \infty$ for all $\tau > \tau_0$. Then

$$\|P_\tau\varphi\|_p \le m\|(A_\tau - \lambda I)P_\tau\varphi\|_p$$

for every $\varphi \in L^p(0,\infty)$ and all $\tau > \tau_0$, whence $\|\varphi\|_p \le m\|(A-\lambda I)\varphi\|_p$ for every $\varphi \in L^p(0,\infty)$. This shows that $A - \lambda I$ has a closed range and a trivial kernel. Our assumption implies that $\|(A_\tau^* + \overline{\lambda} I)^{-1}\|_q \le m < \infty$ where $1/p + 1/q = 1$. Arguing as above, we see that $A^* - \overline{\lambda} I$ has a trivial kernel on $L^q(0,\infty)$, which implies that the range of $A - \lambda I$ is dense in $L^p(0,\infty)$. In summary, $A - \lambda I$ must be invertible on $L^p(0,\infty)$. As this is impossible if $\lambda \in \sigma_p(A)$, we see that (53) is true.

From (53) we deduce the existence of $\tau_1, \tau_2, \ldots$ such that $\tau_n \to \infty$ and

$$\|(A_{\tau_n} - \lambda I)^{-1}\|_p \geq 1/\varepsilon.$$

This shows that λ belongs to the set on the right of (52).

Now suppose $\lambda \in \sigma_p^\varepsilon(A) \setminus \sigma_p(A)$. Then $\|(A - \lambda I)^{-1}\|_p \geq 1/\varepsilon$. Let $U \subset \mathbf{C}$ be any open neighborhood of λ. From Theorem 5.1 we see that there is a point $\mu \in U$ such that $\|(A - \mu I)^{-1}\| > 1/\varepsilon$. Hence, we can find an integer $n_0 > 0$ such that

$$\|(A - \mu I)^{-1}\|_p \geq \frac{1}{\varepsilon - 1/n} \text{ for all } n \geq n_0.$$

Because U was arbitrary, it follows that there exists a sequence $\lambda_1, \lambda_2, \ldots$ such that $\lambda_n \in \sigma_p^{\varepsilon - 1/n}(A)$ and $\lambda_n \to \lambda$.

For every invertible operator $T \in \mathcal{L}(L^p(0,\infty))$ we have

$$\|T^{-1}\|_p = \sup_{\psi \neq 0} \frac{\|T^{-1}\psi\|_p}{\|\psi\|_p} = \sup_{\varphi \neq 0} \frac{\|\varphi\|_p}{\|T\varphi\|_p} = \left(\inf_{\varphi \neq 0} \frac{\|T\varphi\|_p}{\|\varphi\|_p} \right)^{-1}. \tag{54}$$

Since $\|(A - \lambda_n I)^{-1}\|_p \geq 1/(\varepsilon - 1/n)$, we obtain from (54) that

$$\inf_{\|\varphi\|_p = 1} \|(A - \lambda_n I)\varphi\|_p \leq \varepsilon - 1/n,$$

implying the existence of a $\varphi_n \in L^p(0,\infty)$ such that

$$\|\varphi_n\|_p = 1 \text{ and } \|(A - \lambda_n I)\varphi_n\|_p < \varepsilon - 1/(2n).$$

Because $\|(A_\tau - \lambda_n I)P_\tau \varphi_n\|_p \to \|(A - \lambda I)\varphi\|_p$ and $\|P_\tau \varphi_n\|_p \to \|\varphi_n\|_p = 1$ as $\tau \to \infty$, it follows that

$$\|(A_\tau - \lambda_n I)P_\tau \varphi_n\|_p / \|P_\tau \varphi_n\|_p < \varepsilon - 1/(3n)$$

for all $\tau \to \tau_0(n)$. Again invoking (54) we see that

$$\|(A_\tau - \lambda_n I)^{-1}\|_p > \Big(\varepsilon - 1/(3n)\Big)^{-1} > 1/\varepsilon$$

and thus $\lambda_n \in \sigma_p^\varepsilon(A_\tau)$ for all $\tau > \tau_0(n)$. This implies that $\lambda = \lim \lambda_n$ lies in the set on the right of (52). At this point the proof of (52) is complete.

Repeating the above reasoning with $R_\tau A_\tau R_\tau$ and B in place of A_τ and A, respectively, we get the inclusion

$$\sigma_p^\varepsilon(B) \subset \lim_{\tau \to \infty} \sigma_p^\varepsilon(R_\tau A_\tau R_\tau). \tag{55}$$

Because R_τ is an isometry on $L^p(0,\tau)$ and

$$R_\tau A_\tau R_\tau - \lambda P_\tau = R_\tau (A_\tau - \lambda I) R_\tau,$$

it is clear that $\sigma_p^\varepsilon(R_\tau A_\tau R_\tau) = \sigma_p^\varepsilon(A_\tau)$. Thus, in (55) we may replace $R_\tau A_\tau R_\tau$ by A_τ, which in conjunction with (52) proves the inclusion "$\supset$" in (51).

We are left with proving the inclusion "$\subset$" of (51). So let $\lambda \notin \sigma_p^\varepsilon(A) \cup \sigma_p^\varepsilon(B)$. Then

$$\|(A - \lambda I)^{-1}\|_p < 1/\varepsilon, \;\; \|(B - \lambda I)^{-1}\|_p < 1/\varepsilon,$$

whence

$$\|(A_\tau - \lambda I)^{-1}\|_p < 1/\varepsilon - \delta < 1/\varepsilon \text{ for all } \tau > \tau_0 \tag{56}$$

with some $\delta > 0$ due to Corollary 3.2. If $|\mu - \lambda|$ is sufficiently small, then $A_\tau - \mu I$ is invertible together with $A_\tau - \lambda I$, and we have

$$\|(A_\tau - \mu I)^{-1}\|_p \le \frac{\|(A_\tau - \lambda I)^{-1}\|_p}{1 - |\mu - \lambda| \, \|(A_\tau - \lambda I)^{-1}\|_p}. \tag{57}$$

Let $|\mu - \lambda| < \varepsilon\delta/(1/\varepsilon - \delta)$. In this case (56) and (57) give

$$\|(A_\tau - \mu I)^{-1}\|_p < \frac{1/\varepsilon - \delta}{1 - \varepsilon\delta(1/\varepsilon - \delta)/(1/\varepsilon - \delta)} = \frac{1}{\varepsilon}.$$

Thus, $\mu \notin \sigma_p^\varepsilon(A_\tau)$ for $\tau > \tau_0$. This shows that λ cannot belong to $\lim\limits_{\tau\to\infty} \sigma_p^\varepsilon(A_\tau)$. $\square$

Obviously, Theorem 1.4 is a special case of Theorem 5.2. We remark that in the $p = 2$ case the equalities (34) and (35) imply that

$$\lim_{\tau\to\infty} \sigma_2^\varepsilon\Big(W_\tau(a)\Big) = \sigma_2^\varepsilon\Big(W(a)\Big) \tag{58}$$

for each $\varepsilon > 0$. With the notation as in Corollary 3.4, we have for each $\varepsilon > 0$,

$$\lim_{\tau\to\infty} \sigma_p^\varepsilon\left(\sum_j \prod_k W_\tau(a_{jk}\right) = \sigma_p^\varepsilon(A) \cup \sigma_p^\varepsilon(B_1),$$

$$\lim_{\tau\to\infty} \sigma_p^\varepsilon\left(P_\tau \sum_j \prod_k W(a_{jk}) P_\tau\right) = \sigma_p^\varepsilon(A) \cup \sigma_p^\varepsilon(B_2).$$

Finally, note that Theorems 5.2 and 1.2 imply that $\sigma_p^\varepsilon(W_\tau(a))$ in general depends on p.

6. Matrix Case

Given a positive integer m, we denote by $L_m^p(0,\infty)$ the direct sum of m copies of $L^p(0,\infty)$ and we think of the elements of $L_m^p(0,\infty)$ as columns. We define the norm on $L_m^p(0,\infty)$ by

$$\|(f_1, \dots, f_m)^T\|_p^p := \int_0^\infty \left(\sum_{j=1}^m |f_j(x)|^2\right)^{p/2} dx.$$

Let $C_p^{m\times m}(\dot{\mathbf{R}})$ stand for the collection of all $m \times m$ matrix functions with entries in $C_p(\dot{\mathbf{R}})$. For $a = (a_{jk})_{j,k=1}^m \in C_p^{m\times m}(\dot{\mathbf{R}})$ the (block) Wiener-Hopf integral operator $W(a)$ is defined on $L_m^p(0,\infty)$ by the operator matrix $(W(a_{jk}))_{j,k=1}^m$. The direct sum $L_m^p(0,\tau)$ of m copies of $L^p(0,\tau)$ may be identified with a subspace of $L_m^p(0,\infty)$ in a natural manner. The operators on $L_m^p(0,\infty)$ given by the diagonal matrices $\operatorname{diag}(P_\tau, \dots, P_\tau)$ and $\operatorname{diag}(R_\tau, \dots, R_\tau)$ are also denoted by P_τ and R_τ. If $a = (a_{jk})_{j,k=1}^m \in C_p^{m\times m}(\dot{\mathbf{R}})$, then the compression $W_\tau(a)$ of $W(a)$ to $L_m^p(0,\tau)$ is nothing but $(W_\tau(a_{jk}))_{j,k=1}^m$. Finally, we give $\mathcal{F}_p^{m\times m}$, $\mathcal{C}_p^{m\times m}$, $\mathcal{A}_p^{m\times m}$ the natural meaning.

The results proved above extend to the matrix case with obvious adjustments. We confine ourselves to formulating the basic theorems.

Theorem 6.1. *Let* $1 \le p < \infty$. *Then*

$$\mathcal{A}_p^{m\times m} = \Big\{ W_\tau(a) + K : a \in C_p^{m\times m}(\dot{\mathbf{R}}),\ K \in \mathcal{K}\big(L_m^p(0,\infty)\big) \Big\},$$

$$\mathcal{F}_p^{m\times m} = \Big\{ W_\tau(a) + P_\tau K P_\tau + R_\tau L R_\tau + C_\tau : a \in C_p^{m\times m}(\dot{\mathbf{R}}),$$
$$K, L \in \mathcal{K}\big(L_m^p(0,\infty)\big),\ \|C_\tau\|_p \to 0 \ \text{ as } \tau \to \infty \Big\},$$

the map $\mathrm{Sym}_p : \mathcal{F}_p^{m\times m}/\mathcal{C}_p^{m\times m} \to \mathcal{A}_p^{m\times m} \oplus \mathcal{A}_p^{m\times m}$ *sending*

$$\{W_\tau(a) + P_\tau K P_\tau + R_\tau L R_\tau + C_\tau\} + \mathcal{C}_p^{m\times m} \ \text{ to } \ \Big(W(a) + K,\ \ W(\tilde{a}) + L \Big)$$

is an isometric Banach algebra homomorphism, and

$$\limsup_{\tau\to\infty} \|A_\tau\|_p = \lim_{\tau\to\infty} \|A_\tau\|_p$$

for every family $\{A_\tau\} \in \mathcal{F}_p^{m\times m}$.

Theorem 6.2. *Let* $1 \le p < \infty$, $a \in C_p^{m\times m}(\dot{\mathbf{R}})$, *and* $N \in \mathcal{K}(L_m^p(0,\infty))$. *If* $W(a) + N$ *and* $W(\tilde{a})$ *(where* $\tilde{a}(\xi) := a(-\xi)$*) are invertible on* $L_m^p(0,\infty)$ *then the limit* (27) *exists and is equal to* (28). *If one of the operators* $W(a) + N$ *and* $W(\tilde{a})$ *is not invertible on* $L_m^p(0,\infty)$, *then*

$$\limsup_{\tau\to\infty} \|(W_\tau(a) + P_\tau N P_\tau)^{-1}\|_p = \infty.$$

Note that in the scalar case ($m = 1$) the operator $W(\tilde{a})$ is automatically invertible if $W(a) + N$ is invertible. This is no longer true in the matrix case (see e.g., [14]). Also notice that in the matrix case the equality

$$\lim_{\tau\to\infty} \|W_\tau^{-1}(a)\|_2 = \max\Big\{ \|W^{-1}(a)\|_2,\ \|W^{-1}(\tilde{a})\|_2 \Big\}$$

cannot be simplified to (36).

Theorem 6.3. *Let* $1 < p < \infty$ *and* $A_\tau = W_\tau(a) + P_\tau K P_\tau + R_\tau L R_\tau + C_\tau$, *where* $a \in C_p^{m\times m}(\dot{\mathbf{R}})$, $K, L \in \mathcal{K}(L_m^p(0,\infty))$, *and* $\|C_\tau\|_p \to 0$ *as* $\tau \to \infty$. *Then for each* $\varepsilon > 0$,

$$\lim_{\tau\to\infty} \sigma_p^\varepsilon(A_\tau) = \sigma_p^\varepsilon\Big(W(a) + K\Big) \cup \sigma_p^\varepsilon\Big(W(\tilde{a}) + L\Big).$$

We emphasize again that the equality

$$\lim_{\tau\to\infty} \sigma_2^\varepsilon\Big(W_\tau(a)\Big) = \sigma_2^\varepsilon\Big(W(a)\Big) \cup \sigma_2^\varepsilon\Big(W(\tilde{a})\Big)$$

cannot be reduced to (58) if $m > 1$.

7. Block Toeplitz Matrices

Let $\mathbf{T}$ be the complex unit circle. Given a function $a \in L^\infty(\mathbf{T})$, we denote by $\{a_n\}_{n\in\mathbf{Z}}$ the sequence of its Fourier coefficients,

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-in\theta} d\theta,$$

and we let $W(\mathbf{T})$ stand for the Wiener algebra of all functions a with absolutely convergent Fourier series. We think of $l^p = l^p(\mathbf{Z}_+) = l^p(\{0,1,2,\dots\})$ as a space of infinite columns. If $a \in L^\infty(\mathbf{T})$, then the Toeplitz operator $T(a)$ given by the Toeplitz matrix

$$(a_{j-k})_{j,k=0}^{\infty} \tag{59}$$

is bounded on l^2. For $1 \le p < \infty$, let $M_p(\mathbf{T})$ be the set of all functions $a \in L^\infty(\mathbf{T})$ satisfying

$$\|T(a)\varphi\|_p \le C\|\varphi\|_p \text{ for all } \varphi \in l^2 \cap l^p$$

with some $C = C(a,p) < \infty$ independent of p. The set $M_p(\mathbf{T})$ is a Banach algebra with the norm

$$\|a\|_{M_p(\mathbf{T})} = \sup\Big\{\|T(a)\varphi\|_p/\|\varphi\|_p : \varphi \in l^2 \cap l^p,\ \varphi \ne 0\Big\}.$$

It is easily seen that $W(\mathbf{T}) \subset M_p(\mathbf{T})$ for all $p \in [1,\infty)$. The closure of $W(\mathbf{T})$ in $M_p(\mathbf{T})$ is denoted by $C_p(\mathbf{T})$. One can show that $C_1(\mathbf{T}) = W(\mathbf{T})$, $C_2(\mathbf{T}) = C(\mathbf{T})$, and that if $p \in (1,2) \cup (2,\infty)$, then

$$C_p(\mathbf{T}) = C_q(\mathbf{T}) \subset C(\mathbf{T}) \cap M_p(\mathbf{T}),$$

the inclusion being proper (see, e.g., [7], [8]). If $a \in C_p(\mathbf{T})$, then the matrix (59) induces a bounded operator on l^p. This operator is denoted by $T(a)$ and called the Toeplitz operator (or the Toeplitz matrix) with the symbol a.

For $a \in C_p(\mathbf{T})$ and $n \in \mathbf{Z}_+$ we denote by $T_n(a)$ the $(n+1)\times(n+1)$ Toeplitz matrix $(a_{j-k})_{j,k=0}^{n}$. On defining P_n on l^p by

$$(P_n\varphi)_j = \begin{cases} \varphi_j & \text{if} \quad 0 \le j \le n, \\ 0 & \text{if} \quad j > n, \end{cases}$$

we can identify $T_n(a)$ with the compression of $T(a)$ to $l^p(\mathbf{Z}_n) = l^p(\{0,1,\dots,n\})$. The discrete analogue of the operator R_τ given by (15) is the operator R_n defined by

$$(R_n\varphi)_j = \begin{cases} \varphi_{n-j} & \text{if} \quad 0 \le j \le n, \\ 0 & \text{if} \quad j > n. \end{cases}$$

We regard $l^p(\mathbf{Z}_n)$ as a subspace of $l^p = l^p(\mathbf{Z}_+)$. This specifies the norms in $l^p(\mathbf{Z}_n)$ and $\mathcal{L}(l^p(\mathbf{Z}_n))$. The norm and the spectrum of an operator A on $l^p(\mathbf{Z}_n)$ or $l^p(\mathbf{Z}_+)$ are denoted by $\|A\|_p$ and $\sigma_p(A)$, respectively. The ε-pseudospectrum $\sigma_p^\varepsilon(A)$ is defined as the set (3).

Finally, given a positive integer m, we define l^p_m, $l^p_m(\mathbf{Z}_n)$, $C_p^{m\times m}(\mathbf{T})$, P_n, R_n as well as $T(a)$ and $T_n(a)$ for $a \in C_p^{m\times m}(\mathbf{T})$ in the natural fashion. The norm of $(f_1,\dots,f_m)^T \in l^p_m$ is defined by

$$\|(f_1,\dots,f_m)^T\|_p^p = \sum_j \left(\sum_{k=1}^m |(f_k)_j|^2\right)^{p/2}.$$

All results established above for Wiener-Hopf integral operators have analogues for Toeplitz operators. The proofs in the Toeplitz case are completely analogous to (and sometimes even simpler than) the proofs for Wiener-Hopf integral operators. We therefore only state the results.

Let $\mathcal{A}_p^{m\times m}$ be the smallest closed subalgebra of $\mathcal{L}(l_m^p)$ containing $\{T(a) : a \in W(\mathbf{T})\}$. One can show that

$$\mathcal{A}_p^{m\times m} = \Big\{ T(a) + K : a \in C_p^{m\times m}(\mathbf{T}),\ K \in \mathcal{K}(l_m^p) \Big\}$$

and that $\mathcal{A}_p^{m\times m}$ is inverse closed in $\mathcal{L}(l_m^p)$. Denote by $\mathcal{S}_p^{m\times m}$ the Banach algebra of all sequences $\{A_n\} = \{A_n\}_{n=0}^{\infty}$ of operators (matrices) $A_n \in \mathcal{L}(l_m^p)$ such that

$$\|\{A_n\}\|_p := \sup_{n\in\mathbf{Z}_+} \|A_n\|_p < \infty,$$

let $\mathcal{C}_p^{m\times m}$ be the closed two-sided ideal of all sequences $\{A_n\} \in \mathcal{S}_p^{m\times m}$ for which $\|A_n\|_p \to 0$ as $n \to \infty$, and let $\mathcal{F}_p^{m\times m}$ stand for the smallest closed subalgebra of $\mathcal{S}_p^{m\times m}$ containing all sequences $\{T_n(a)\}_{n=0}^{\infty}$ with $a \in W(\mathbf{T})$.

Theorem 7.1. *Let $1 \le p < \infty$. Then*

$$\mathcal{F}_p^{m\times m} = \Big\{ T_n(a) + P_nKP_n + R_nLR_n + C_n : a \in C_p^{m\times m}(\mathbf{T}),\\ K, L \in \mathcal{K}(l_m^p),\ \|C_n\|_p \to 0 \ \textit{as}\ n \to \infty \Big\}.$$

The map

$$\mathrm{Sym}_p : \mathcal{F}_p^{m\times m}/\mathcal{C}_p^{m\times m} \to \mathcal{A}_p^{m\times m} \oplus \mathcal{A}_p^{m\times m}$$

sending

$$\{A_n\} + \mathcal{C}_p^{m\times m} = \Big\{ T_n(a) + P_nKP_n + R_nLR_n + C_n \Big\} + \mathcal{C}_p^{m\times m}$$

to

$$\Big(\mathrm{s}\!-\!\lim_{n\to\infty} A_n,\ \mathrm{s}\!-\!\lim_{n\to\infty} R_nA_nR_n \Big) = \Big(T(a) + K,\ T(\tilde{a}) + L \Big)$$

($\tilde{a}(t) := a(1/t)$ for $t \in \mathbf{T}$) is an isometric Banach algebra homomorphism. Furthermore,

$$\limsup_{n\to\infty} \|A_n\|_p = \lim_{n\to\infty} \|A_n\|_p$$

for every $\{A_n\} \in \mathcal{F}_p^{m\times m}$.

Theorem 7.2. *Let $1 \le p < \infty$, $a \in C_p^{m\times m}(\mathbf{T})$, and $N \in \mathcal{K}(l_m^p)$. If $T(a) + N$ and $T(\tilde{a})$ are invertible on l_m^p then the limit*

$$\lim_{n\to\infty} \|(T_n(a) + P_nNP_n)^{-1}\|_p$$

exists and equals

$$\max\Big\{ \|(T(a)+N)^{-1}\|_p,\ \|T^{-1}(\tilde{a})\|_p \Big\}.$$

If one of the operators $T(a) + N$ and $T(\tilde{a})$ is not invertible on l_m^p then

$$\limsup_{n\to\infty} \|(T_n(a) + P_nNP_n)^{-1}\|_p = \infty.$$

Theorem 7.3. *Let $1 < p < \infty$ and $A_n = T_n(a) + P_n K P_n + R_n L R_n + C_n$ where $a \in C_p^{m\times m}(\mathbf{T})$, $K, L \in \mathcal{K}(l_m^p)$, and $\|C_N\|_p \to 0$ as $n \to \infty$. Then for each $\varepsilon > 0$,*

$$\lim_{n\to\infty} \sigma_p^\varepsilon(A_n) = \sigma_p^\varepsilon\Big(T(a) + K\Big) \cup \sigma_p^\varepsilon\Big(T(\tilde{a}) + L\Big).$$

In Theorem 7.3, $\lim_{n\to\infty} \sigma_p^\varepsilon(A_n)$ is defined as the set of all $\lambda \in \mathbf{C}$ for which there exist $n_1, n_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ such that

$$n_1 < n_2 < \ldots,\ n_k \to \infty,\ \lambda_n \in \sigma_p^\varepsilon(A_{n_k}),\ \lambda_n \to \lambda.$$

For $p = 2$, the previous three theorems were established in [31] and [4] by means of C^*-algebra techniques. For $p = 1$, $m = 1$, $N = 0$, Theorem 7.2 was first obtained in [16]. The paper [16] also contains an example of a function $a \in W(\mathbf{T})$ such that

$$\|T^{-1}(a)\|_1 > \|T^{-1}(a)\|_\infty; \tag{60}$$

the proof of Theorem 1.2 given here is nothing but the continuous analogue of the proof of (60) presented in [16]. Of course, once (60) is known, one can proceed as in the proof of Theorem 1.2 to show that

$$\|T^{-1}(a)\|_p < \|T^{-1}(\tilde{a})\|_p$$

if only p is sufficiently large. We remark that in [16] one can also find estimates for the convergence speed of $\|T_n^{-1}(a)\|_1$ to $\|T^{-1}(a)\|_1$ as $n \to \infty$.

The discrete analogue of Theorem 4.1 amounts to the triviality that the eigenvalues of $T_n(a)$ do not depend on the space $T_n(a)$ is thought of as acting on.

Given $a, b \in C_p^{m\times m}(\mathbf{T})$, we put

$$aP + bQ := \left(\begin{array}{ccc|cccc} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & b_0 & b_{-1} & a_{-2} & a_{-3} & a_{-4} & \ldots \\ \ldots & b_1 & b_0 & a_{-1} & a_{-2} & a_{-3} & \ldots \\ \hline \ldots & b_2 & b_1 & a_0 & a_{-1} & a_{-2} & \ldots \\ \ldots & b_3 & b_2 & a_1 & a_0 & a_{-1} & \ldots \\ \ldots & b_4 & b_3 & a_2 & a_1 & a_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right) \tag{61}$$

and call $aP + bQ$ a paired matrix (see [14, Chapters V and VI]). Let $l_m^p(\mathbf{Z})$ $(1 \le p < \infty)$ denote the usual l_m^p spaces on the integers $\mathbf{Z}$ and define P_n on $l_m^p(\mathbf{Z})$ by

$$P_n\ :\ (x_k)_{k=-\infty}^{\infty} \mapsto (\ldots, 0, x_{-n}, \ldots, x_{-1}, x_0, \ldots, x_{n-1}, 0, \ldots).$$

If $K \in \mathcal{K}(l_m^p(\mathbf{Z}))$, then $P_n(aP + bQ + K)P_n$ may be identified with a $2nm \times 2nm$ matrix. The hypothesis $a, b \in C_p^{m\times m}(\mathbf{T})$ implies that the north-east and south-west quarters of the matrix (61) represent compact operators. Thus (61) is in fact a compactly perturbed direct sum of two Toeplitz matrices. From Theorems 7.2 and 7.3 we therefore easily deduce that

$$\begin{aligned} &\lim_{n\to\infty} \|(P_n(aP + bQ + K)P_n)^{-1}\|_p \\ &\qquad = \max\Big\{\|(aP + bQ + K)^{-1}\|_p, \|T^{-1}(\tilde{a})\|_p, \|T^{-1}(\tilde{b})\|_p\Big\} \end{aligned}$$

and, for $\varepsilon > 0$,

$$\lim_{n\to\infty} \sigma_p^\varepsilon\Big(P_n(aP + bQ + K)P_n\Big) = \sigma_p^\varepsilon\Big(aP + bQ + K\Big) \cup \sigma_p^\varepsilon\Big(T(\tilde{a})\Big) \cup \sigma_p^\varepsilon\Big(T(\tilde{b})\Big).$$

Of course, systems of paired Wiener-Hopf integral operators may be tackled in the same way.

We remark that all results can also be carried over to block Toeplitz operators on the Hardy spaces $H^p(\mathbf{T})$ $(1 < p < \infty)$ and to Wiener-Hopf integral operators with matrix symbols on the Hardy spaces $H^p(\mathbf{R})$ $(1 < p < \infty)$.

8. Pseudospectra of Infinite Toeplitz Matrices

From (6) we know the spectrum of the Wiener-Hopf operator $W(a)$ on $L^p(0,\infty)$ $(1 \le p \le \infty)$ in case $a \in \mathbf{C} + FL^1(\mathbf{R})$. An analogous result is true for Toeplitz operators on l^p $(1 \le p \le \infty)$: if $b \in W(\mathbf{T})$, then

$$\sigma_p\Big(T(b)\Big) = b(\mathbf{T}) \cup \Big\{\lambda \in \mathbf{C} \setminus b(\mathbf{T}) : \operatorname{wind}(b,\lambda) \ne 0\Big\}. \tag{62}$$

Theorems 5.2 and 7.3 do not pertain to spectra but to pseudospectra. Thus, it is desirable to know more about the pseudospectra $\sigma_p^\varepsilon(W(a))$ and $\sigma_p^\varepsilon(T(b))$.

Given a Banach space X and a bounded linear operator A on X, we define the spectrum $\sigma_X(A)$ and the ε-pseudospectrum $\sigma_X^\varepsilon(A)$ in the natural manner:

$$\sigma_X(A) = \Big\{\lambda \in \mathbf{C} : A - \lambda I \text{ is not invertible}\Big\},$$
$$\sigma_X^\varepsilon(A) = \Big\{\lambda \in \mathbf{C} : \|(A-\lambda I)^{-1}\| \ge 1/\varepsilon\Big\}.$$

One always has

$$\sigma_X(A) + \Delta_\varepsilon \subset \sigma_X^\varepsilon(A) \tag{63}$$

where $\Delta_\varepsilon = \{\lambda \in \mathbf{C} : |\lambda| \le \varepsilon\}$. Indeed, if $\lambda \notin \sigma_X^\varepsilon(A)$ then $\varepsilon < \|(A-\lambda I)^{-1}\|^{-1}$, which implies that $A - \lambda I - \delta I$ is invertible whenever $|\delta| \le \varepsilon$.

The following two theorems provide additional information about Hilbert space Wiener-Hopf and Toeplitz operators. For a function c in L^∞ on $\mathbf{R}$ or $\mathbf{T}$, we denote by $\mathcal{R}(c)$ its essential range and by $\operatorname{conv}\mathcal{R}(c)$ the convex hull of $\mathcal{R}(c)$. Note that $\sigma_X^0(A)$ is nothing but $\sigma_X(A)$.

Theorem 8.1. *If* $\varepsilon \ge 0$, $a \in L^\infty(\mathbf{R})$, $b \in L^\infty(\mathbf{T})$, *then*

$$\sigma_2\Big(W(a)\Big) + \Delta_\varepsilon \subset \sigma_2^\varepsilon\Big(W(a)\Big) \subset \operatorname{conv}\mathcal{R}(a) + \Delta_\varepsilon, \tag{64}$$

$$\sigma_2\Big(T(b)\Big) + \Delta_\varepsilon \subset \sigma_2^\varepsilon\Big(T(b)\Big) \subset \operatorname{conv}\mathcal{R}(b) + \Delta_\varepsilon. \tag{65}$$

Proof. The left inclusions in (64) and (65) are immediate from (63). For $\varepsilon = 0$, the right inclusions of (64) and (65) are known as the Brown-Halmos theorem (see, e.g., [8, Theorem 2.33]). For $\varepsilon > 0$, the proof of the latter inclusions is similar to the proof of the Brown-Halmos theorem. Here it is.

Let $\lambda \notin \operatorname{conv}\mathcal{R}(a) + \Delta_\varepsilon$. Then $0 \notin \operatorname{conv}\mathcal{R}(a-\lambda) + \Delta_\varepsilon$, and hence we can rotate $\operatorname{conv}\mathcal{R}(a-\lambda) + \Delta_\varepsilon$ into the right open half-plane, i.e., there is a $\gamma \in \mathbf{T}$ such that

$$\operatorname{conv}\mathcal{R}\Big(\gamma(a-\lambda)\Big) + \Delta_\varepsilon \subset \{z \in \mathbf{C} : \operatorname{Re} z > 0\}.$$

Then we can find a (sufficiently large) disk $\{z \in \mathbf{C} : |z - r - \varepsilon| < r\}$ containing the set $\operatorname{conv} \mathcal{R}\Big(\gamma(a-\lambda)\Big)$. We so have $|\gamma(a(\xi)-\lambda) - r - \varepsilon| < r$ and therefore

$$\left| \frac{\gamma(a(\xi)-\lambda)}{r+\varepsilon} - 1 \right| < \frac{r}{r+\varepsilon}$$

for almost all $\xi \in \mathbf{R}$. Since

$$\frac{\gamma}{r+\varepsilon} W(a-\lambda) = I + W\left(\frac{\gamma(a-\lambda)}{r+\varepsilon} - 1\right)$$

and the norm of the Wiener-Hopf operator on the right is less than $r/(r+\varepsilon) < 1$, it follows that $(\gamma/(r+\varepsilon))W(a-\lambda)$ is invertible and that

$$\frac{r+\varepsilon}{|\gamma|} \|W^{-1}(a-\lambda)\|_2 < \frac{1}{1 - r/(r+\varepsilon)},$$

whence $\|W^{-1}(a-\lambda)\|_2 < |\gamma|/\varepsilon = 1/\varepsilon$. Thus, $\lambda \notin \sigma_p^\varepsilon(W(a))$.

The proof is the same for the Toeplitz operator $T(b)$. □

It is readily seen that the right inclusions of (64) and (65) may be proper. Indeed, suppose $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $\mathcal{R}(a)$ is the half-circle $\{z \in \mathbf{T} : \operatorname{Im} z \geq 0\}$. From (6) we infer that $W(a)$ is invertible and hence, if $\varepsilon > 0$ is small enough then $\|W^{-1}(a)\|_2 < 1/\varepsilon$. Thus, $0 \notin \sigma_2^\varepsilon(W(a))$ although $0 \in \operatorname{conv} \mathcal{R}(a) + \Delta_\varepsilon$. The following result tells us that the left inclusions of (64) and (65) may also be proper.

Theorem 8.2. *Given $\varepsilon > 0$, there exist $a \in \mathbf{C} + FL^1(\mathbf{R})$ and $b \in W(\mathbf{T})$ such that*

$$\sigma_2\Big(W(a)\Big) + \Delta_\varepsilon \neq \sigma_2^\varepsilon\Big(W(a)\Big), \quad \sigma_2\Big(T(b)\Big) + \Delta_\varepsilon \neq \sigma_2^\varepsilon\Big(T(b)\Big).$$

Proof. Put

$$b(e^{i\theta}) = \begin{cases} e^{2i\theta} & \text{for} \quad 0 \leq \theta < \pi, \\ e^{-2i\theta} & \text{for} \quad \pi \leq \theta < 2\pi. \end{cases}$$

If $e^{i\theta}$ traverses $\mathbf{T}$, then $b(e^{i\theta})$ twice traces out the unit circle, once in the positive and once in the negative direction. Thus, $\sigma_2(T(b)) = \mathbf{T}$ due to (62). We claim that

$$\sigma_2^{3/4}\Big(T(b)\Big) = \{\lambda \in \mathbf{C} : |\lambda| \leq 7/4\}, \tag{66}$$

which is clearly properly larger than $\mathbf{T} + \Delta_{3/4}$. From Theorem 8.1 we deduce that

$$\{\lambda \in \mathbf{C} : 1/4 \leq |\lambda| \leq 7/4\} \subset \sigma_2^{3/4}\Big(T(b)\Big) \subset \{\lambda \in \mathbf{C} : |\lambda| \leq 7/4\}.$$

Consequently, (66) will follow as soon as we have shown that

$$\|T^{-1}(b-\lambda)\|_2 \geq 4/3 \ \text{ whenever } \ |\lambda| < 1/4. \tag{67}$$

So assume $|\lambda| < 1/4$. The Fourier coefficients of b can be easily computed:

$$\begin{aligned} &b_0 = 0, \ b_2 = b_{-2} = 1/2, \\ &b_n = 0 \ \text{ if } \ n \neq \pm 2 \ \text{ is even}, \\ &b_n = 4/\Big(\pi i (n^2 - 4)\Big) \ \text{ if } \ n \ \text{ is odd}. \end{aligned}$$

It follows in particular that $b \in W(\mathbf{T})$. Writing the elements of l^2 in the form $(x_0, x_1, x_2, \dots)$ and taking into account (54) we get

$$\begin{aligned}
\|T^{-1}(b-\lambda)\|_2 &= \left(\inf_{\varphi \neq 0} \frac{\|T(b-\lambda)\varphi\|_2}{\|\varphi\|_2} \right)^{-1} \\
&\geq \|T(b-\lambda)(1,0,0,\dots)\|_2^{-1} \\
&= \|(b_0-\lambda, b_1, b_2, b_3, \dots)\|_2^{-1} \\
&= \left(|b_0-\lambda|^2 + \sum_{n\geq 1} |b_n|^2 \right)^{-1/2}.
\end{aligned}$$

Since $b_0 = 0$ and $|b_n| = |b_{-n}|$ for all n, we obtain

$$\begin{aligned}
|b_0-\lambda|^2 + \sum_{n\geq 1} |b_n|^2 &= |\lambda|^2 + \frac{1}{2}\sum_{n\in\mathbf{Z}} |b_n|^2 \\
&= |\lambda|^2 + \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} |b(e^{i\theta})|^2 d\theta \quad \text{(Parseval's equality)} \\
&= |\lambda|^2 + 1/2 \quad \text{(note that } b \text{ is unimodular)} \\
&< (1/4)^2 + 1/2 = 9/16.
\end{aligned}$$

Thus, $\|T^{-1}(b-\lambda)\|_2 > (9/16)^{-1/2} = 4/3$, which gives (67) and completes the proof of (66).

Replacing b by $b_\varepsilon := (4\varepsilon/3)b$ we easily conclude from (66) that

$$\sigma_2^\varepsilon\Big(T(b_\varepsilon)\Big) = \{\lambda \in \mathbf{C} : |\lambda| \leq 7\varepsilon/3\},$$

which is strictly larger than

$$\sigma_2\Big(T(b_\varepsilon)\Big) + \Delta_\varepsilon = \{\lambda \in \mathbf{C} : \varepsilon/3 \leq |\lambda| \leq 7/\varepsilon\}.$$

At this point we have proved the assertion for Toeplitz matrices.

To prove the theorem for Wiener-Hopf operators, notice first that an orthonormal basis in $L^2(0,\infty)$ is given by $\{e_n\}_{n=0}^\infty$ where $e_n(x) = \sqrt{2}e^{-x}\Pi_n(2x)$ and Π_n is the normalized nth Laguerre polynomial. The map

$$U : l^2 \to L^2(0,\infty),\ (\varphi_0, \varphi_1, \varphi_2, \dots) \mapsto \sum_{n=0}^\infty \varphi_n e_n$$

is an isometric isomorphism, and if $b \in W(\mathbf{T})$ has the Fourier coefficients $\{b_n\}_{n\in\mathbf{Z}}$ then $UT(b)U^{-1}$ is the Wiener-Hopf operator $W(a)$ with the symbol

$$a(\xi) = \sum_{n\in\mathbf{Z}} b_n \left(\frac{\xi - i}{\xi + i}\right)^n \quad (\xi \in \mathbf{R});$$

see, e.g., [14, Chap. III, Sec. 3]. Obviously, $a \in \mathbf{C} + FL^1(\mathbf{R})$, $\sigma_2(W(a)) = \sigma_2(T(b))$, and since

$$\|W^{-1}(a-\lambda)\|_2 = \|UT^{-1}(b-\lambda)U^{-1}\|_2 = \|T^{-1}(b-\lambda)\|_2,$$

it results that $\sigma_2^\varepsilon(W(a)) = \sigma_2^\varepsilon(T(b))$ for each $\varepsilon > 0$. This reduces the proof in the Wiener-Hopf case to the proof in the Toeplitz case. □

References

[1] P.M. Anselone and I.H. Sloan, *Spectral approximations for Wiener-Hopf operators* II, J. Integral Equations Appl. **4** (1992), 465–489.

[2] E. Basor and K.E. Morrison, *The Fisher-Hartwig conjecture and Toeplitz eigenvalues*, Linear Algebra Appl. **202** (1994), 129–142.

[3] A. Böttcher, *Wiener-Hopf determinants with rational symbols*, Math. Nachr. **144** (1989), 39–64.

[4] A. Böttcher, *Pseudospectra and singular values of large convolution operators*, J. Integral Equations Appl. **6** (1994), 267–301.

[5] A. Böttcher, *Infinite matrices and projection methods*, Lectures on Operator Theory and Its Applications (P. Lancaster, ed.), Fields Institute Monographs, no. 3, Amer. Math. Soc., Providence, R.I., 1996, pp. 1–72.

[6] A. Böttcher and B. Silbermann, *The finite section method for Toeplitz operators on the quarter-plane with piecewise continuous symbols*, Math. Nachr. **110** (1983), 279–291.

[7] A. Böttcher and B. Silbermann, *Invertibility and Asymptotics of Toeplitz Matrices*, Akademie-Verlag, Berlin, 1983.

[8] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Akademie-Verlag, Berlin, 1989 and Springer-Verlag, Berlin, Heidelberg, New York, 1990.

[9] A. Böttcher and H. Widom, *Two remarks on spectral approximations for Wiener–Hopf operators*, J. Integral Equations Appl. **6** (1994), 31–36.

[10] A. Böttcher and H. Wolf, *Spectral approximation for Segal-Bargmann space Toeplitz operators*, Linear Operators, Banach Center Publ., no. 38, Polish Acad. Sci., Warsaw (to appear).

[11] K.M. Day, *Toeplitz matrices generated by the Laurent series expansion of an arbitrary rational function*, Trans. Amer. Math. Soc. **206** (1975), 224–245.

[12] K.M. Day, *Measures associated with Toeplitz matrices generated by Laurent expansion of rational functions*, Trans. Amer. Math. Soc. **209** (1975), 175–183.

[13] R. Duduchava, *Integral Equations with Fixed Singularities*, Teubner-Verlag, Leipzig, 1979.

[14] I. Gohberg and I.A. Feldman, *Convolution Equations and Projection Methods for Their Solution*, Amer. Math. Soc., Providence, R.I., 1974 (Russian original: Nauka, Moscow, 1971; German translation: Akademie-Verlag, Berlin, 1974).

[15] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, Univ. Calif. Press, Berkeley and Los Angeles, 1958.

[16] S.M. Grudsky and A.V. Kozak, *On the convergence speed of the norms of the inverses of truncated Toeplitz operators*, Integro-Differential Equations and Their Applications, Rostov-on-Don Univ. Press, 1995, pp. 45–55 (in Russian).

[17] R. Hagen, S. Roch, B. Silbermann, *Spectral Theory of Approximation Methods for Convolution Equations*, Birkhäuser Verlag, Basel, Boston, Berlin, 1995.

[18] A.V. Kozak and I.B. Simonenko, *Projection methods for the solution of multidimensional discrete convolution equations*, Sibir. Mat. Zh. **21** (1980), 119–127 (in Russian).

[19] M.G. Krein, *Integral equations on a half-line with kernel depending upon the difference of the arguments*, Amer. Math. Soc. Transl. (2) **22** (1962), 163–288 (Russian original: Uspekhi Mat. Nauk **13** (1958), 3–120).

[20] H. Landau, *On Szegö's eigenvalue distribution theorem and non-Hermitian kernels*, J. d'Analyse Math. **28** (1975), 335–357.

[21] H. Landau, *Loss in unstable resonators*, J. Opt. Soc. Am. **66** (1976), 525–529.

[22] H. Landau, *The notion of approximate eigenvalues applied to an integral equation of laser theory*, Quaterly Appl. Math., April 1977, 165–171.

[23] K.E. Morrison, *Spectral approximation of multiplication operators*, New York J. Math. **1** (1995), 75–96.

[24] S. Prössdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations*, Akademie-Verlag, Berlin, 1991, and Birkhäuser Verlag, Basel, Boston, Berlin 1991.

[25] S.C. Reddy, *Pseudospectra of Wiener-Hopf integral operators and constant-coefficient differential operators*, J. Integral Equations Appl. **5** (1993), 369–403.

[26] L. Reichel and L.N. Trefethen, *Eigenvalues and pseudo-eigenvalues of Toeplitz matrices*, Linear Algebra Appl. **162** (1992), 153–185.

[27] S. Roch and B. Silbermann, *Limiting sets of eigenvalues and singular values of Toeplitz matrices*, Asymptotic Analysis **8** (1994), 293–309.
[28] S. Roch and B. Silbermann, *C^*-algebra techniques in numerical analysis*, J. Operator Theory **35** (1996), 241–280.
[29] P. Schmidt and F. Spitzer, *The Toeplitz matrices of an arbitrary Laurent polynomial*, Math. Scand. **8** (1960), 15–38.
[30] B. Silbermann, *Lokale Theorie des Reduktionsverfahrens für Toeplitzoperatoren*, Math. Nachr. **104** (1982), 45–56.
[31] B. Silbermann, *On the limiting set of singular values of Toeplitz matrices*, Linear Algebra Appl. **182** (1993), 35–43.
[32] B. Silbermann, *Asymptotic invertibility of Toeplitz operators*, Operator Theory: Advances and Applications **90** (1996), 295–303.
[33] E.E. Tyrtyshnikov, *A unifying approach to some old and new theorems on distribution and clustering*, Linear Algebra Appl. **232** (1996), 1–43.
[34] H. Widom, *Asymptotic behavior of block Toeplitz matrices and determinants* II, Adv. Math. **21** (1976), 1–29.
[35] H. Widom, *Asymptotic Expansions for Pseudodifferential Operators on Bounded Domains*, Springer-Verlag, New York, 1985.
[36] H. Widom, *Eigenvalue distribution of nonselfadjoint Toeplitz matrices and the asymptotics of Toeplitz determinants in the case of nonvanishing index*, Operator Theory: Advances and Applications **48** (1990), 387–421.

Faculty of Mathematics, TU Chemnitz-Zwickau, 09107 Chemnitz, Germany
aboettch@mathematik.tu-chemnitz.de

Faculty of Mechanics and Mathematics, Rostov-on-Don State University, Bolshaya Sadovaya 105, 344 711 Rostov-on-Don, Russia
grudsk@ns.unird.ac.ru

Faculty of Mathematics, TU Chemnitz-Zwickau, 09107 Chemnitz, Germany
silbermn@mathematik.tu-chemnitz.de