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Convergence of the p-Series for Stationary Sequences

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Abstract. Let (X_n) be a stationary sequence. We prove the following
(i) If the variables (X_n) are iid and $\mathbb{E}(|X_1|) < \infty$ then

$$\lim_{p\to 1+}\left((p-1)\left(\sum_{n=1}^{\infty}\frac{|X_n(x)|^p}{n^p}\right)\right)^{1/p} = \mathbb{E}(|X_1|), a.e.$$

(ii) If $X_n(x) = f(T^n x)$ where $(X, \mathcal{F}, \mu, T)$ is an ergodic dynamical system, then

$$\lim_{p\to 1+}\left((p-1)\left(\sum_{n=1}^{\infty}\left(\frac{f(T^n x)}{n}\right)^p\right)\right)^{1/p} = \int f d\mu \qquad \text{a.e. for } f \geq 0,\ f \in L \log L.$$

Furthermore the maximal function,

$$\sup_{1<p<\infty}(p-1)^{1/p}\left(\sum_{n=1}^{\infty}\left(\frac{f(T^n x)}{n}\right)^p\right)^{1/p} \text{ is integrable for functions,} f \geq 0,\ f \in L \log L.$$

These limits are linked to the maximal function $N^*(x) = \|(\frac{X_n(x)}{n})\|_{1,\infty}$.

Contents

1. Introduction

Let Z_n be a sequence of independent, identically distributed random variables and (a_n) a sequence of positive real numbers. The a.e. convergence of the weighted averages

$$(*) \qquad \frac{\sum_{n=1}^{N} a_n Z_n}{A_n},$$

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where $A_n = \sum_{n=1}^{N} a_n$, has been characterized by B. Jamison, S. Orey and W. Pruitt ([JOP]). They proved that the condition

$$\sup_n \frac{\tilde{N}_n}{n} < \infty \tag{0}$$

where $\tilde{N}_n = \#\{k : \frac{a_k}{A_k} \geq \frac{1}{n}\}$ is necessary and sufficient for the a.e. convergence of the weighted averages $(*)$ to $\mathbb{E}(Z_1)$. In [A1], interested by the a.e. convergence (y) of averages of the form

$$\frac{\sum_{n=1}^{N} X_n(x) g(S^n y)}{N},$$

we considered the maximal function $N^*(x) = \sup_n \frac{N_n(x)}{n}$ where $N_n(x) = \#\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\}$, $(X_k \geq 0)$. We proved the following:

(1) If X_n are iid random variables and $\mathbb{E}(|X_1|) < \infty$ then $N^*(x)$ is finite a.e.

(2) If the X_n are given by an ergodic dynamical system (i.e., $X_n(x) = f(T^n x)$ where $(X, \mathcal{F}, \mu, T)$ is an ergodic dynamical system and f a measurable non-negative function) then for all p, $1 < p < \infty$ there exists a finite constant C_p such that

$$\mu\{x : N^*(x) > \lambda\} \leq \frac{C_p}{\lambda^p} \int |f|^p d\mu \qquad \text{for all } \lambda > 0. \tag{$**$}$$

Furthermore for all p, $1 < p < \infty$, for all $f \in L^p_+$ we have $\lim_{n\to\infty} \frac{N_n(x)}{n} = \int f d\mu$ a.e.

(A closer inspection of the proof of $(**)$ shows that the constant C_p is of the form $\frac{C}{p-1}$ where C is an absolute constant independent of p.)

If $0 < p < \infty$, and $(x_i)_{i\geq 1}$ is a sequence of nonnegative real numbers, set

$$\|(x_i)\|_{p,\infty} = \left(\sup_{\lambda>0} \lambda^p \#\{i \geq 1; |x_i| > \lambda\}\right)^{1/p}.$$

It is easily seen that for $r < p$

$$\|(x_i)\|_{p,\infty} \leq \left(\sum_i |x_i|^p\right)^{1/p} \leq \left(\frac{p}{p-r}\right)^{1/p} \|(x_i)\|_{r,\infty}$$

(cf. [SW]). In particular, for all p, $1 < p \leq 2$ we have

$$(p-1)^{1/p} \left(\sum_i |x_i|^p\right)^{1/p} \leq p^{1/p} \|(x_i)\|_{1,\infty}. \tag{3}$$

As $\|(x_i)\|_{1,\infty} \sim \sup_n \frac{\#\{k: x_k \geq 1/n\}}{n}$, for bounded sequences the previous inequality applied pointwise to a stationary sequence (X_n) of integrable functions gives us not only the existence of the p-series

$$(p-1)^{1/p} \left(\sum_i |\frac{X_i(x)}{i}|^p\right)^{1/p} \qquad \text{for all } p,\ 1 < p \leq \infty,$$

but also the inequality

$$\sup_{1<p<\infty} (p-1)^{1/p} \Big(\sum_i |\frac{X_i(x)}{i}|^p \Big)^{1/p} \leq 2\|(\frac{X_i(x)}{i})\|_{1,\infty} \tag{4}$$

if $\|(\frac{X_i(x)}{i})\|_{1,\infty} < \infty$. The inequality (4) and some of our previous results suggest the study of the limit when p tends to 1^+ of the series

$$(p-1)^{1/p} \Big(\sum_{i=1}^{\infty} |\frac{X_i(x)}{i}|^p \Big)^{1/p}.$$

Definition. Let (X_n) be a stationary sequence of integrable functions. The p series associated to this sequence is the a.e. series (when it exists):

$$(p-1)^{1/p} \Big(\sum_{i=1}^{\infty} |\frac{X_i(x)}{i}|^p \Big)^{1/p}.$$

In this note, using an elementary lemma on sequence of real numbers, we will show that for (X_n) iid with $\mathbb{E}(|X_1|) < \infty$ the p-series

$$(p-1)^{1/p} \Big(\sum_{i=1}^{\infty} |\frac{X_i(x)}{i}|^p \Big)^{1/p} \qquad \text{converges a.e. to } \mathbb{E}(|X_1|)$$

when p tends to 1^+.

The same argument shows that the p series

$$(p-1)^{1/p} \Big(\sum_{i=1}^{\infty} \Big| \frac{\prod_{j=1}^{H} X_{j,i}(x_j)}{i} \Big|^p \Big)^{1/p} \qquad \text{converges a.e. to } \prod_{j=1}^{H} \mathbb{E}(|X_{j,1}|)$$

where $(X_{j,n})_n$ are iid random variables satisfying the condition $\mathbb{E}(|X_{j,1}|) < \infty$, and the variables x_j are selected in a universal way specified in [A1].

We can remark that for each p the function $G_p(x) = (p-1)^{1/p} \Big(\sum_{i=1}^{\infty} |\frac{X_i(x)}{i}|^p \Big)^{1/p}$ is not *integrable*, as $G_p(x) \geq (p-1)^{1/p} \sup_i |\frac{X_i(x)}{i}|$, and for (X_i) iid with $\mathbb{E}(|X_1| \log |X_1|) = \infty$, the function $\sup_i |\frac{X_i(x)}{i}|$ is not integrable, as shown by D. Burkholder in [B]. So $F^*(x) = \sup_{1<p<\infty} G_p(x)$ is a supremum of nonintegrable functions. This makes the handling of the function $F^*(x)$ somewhat delicate.

In the second part of this note we will focus on the ergodic stationary case. We will consider an ergodic dynamical system $(X, \mathcal{F}, \mu, T)$ and a nonnegative measurable function f. Using (2) we will show first that

$$\frac{N_n(f)(x)}{n} = \frac{\#\{k : \frac{f(T^k x)}{k} \geq 1/n\}}{n}$$

converges in L^1 norm to $\int f d\mu$. Then using extrapolation methods we will show that

$$\left\| \left\| \left(\frac{f(T^k x)}{k} \right) \right\|_{1,\infty} \right\|_1 < \infty \quad \text{for} f \in L(\text{Log L}). \tag{5}$$

One of our interests in (5) lies in the following observation: If we denote by $\frac{f(T^{n^*} x)}{n^*}$ a decreasing rearrangement of the sequence $\frac{f(T^n x)}{n}$, then we have

$$\left\| \left(\frac{f(T^k x)}{k} \right) \right\|_{1,\infty} = \sup_n n \frac{f(T^{n^*} x)}{n^*}. \tag{6}$$

Hence for $f \in L(\text{Log L})$, (6) provides us with some information on the decreasing rate of the sequence $\frac{f(T^n x)}{n}$.

Using (5), we will prove that for $f \in L \log L$, $f \geq 0$,

$$M_1^*(x) = \sup_{1<p<\infty} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p}, \tag{6'}$$

and

$$\lim_{p \to 1^+} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} = \int f d\mu \text{ a.e., } (\mu). \tag{7}$$

The integrability of $M_1^*(x)$ for f in LLogL extends the results on the integrability of the $sup_n \frac{f(T^n x)}{n}$ in the ergodic case. We do not know at the present time if (7) holds for $f \in L^1$. Finally, in the third part of this paper we will study the connection between the maximal operators

$$M_1^*(f)(x) = \sup_{1<p<\infty} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p}, \; M_2^*(f)(x) = \sup_N \frac{1}{N} \sum_{n=1}^{N} f(T^n x)$$

and

$$\left\| \left(\frac{f(T^n x)}{n} \right) \right\|_{1,\infty} = N^*(f)(x).$$

If there is no ambiguity we will simply denote these maximal functions by $M_1^*(x)$, $M_2^*(x)$ and $N^*(x)$.

2. Convergence of the p-series for iid sequences

2.1. The one dimensional case. The next elementary lemma will be useful for the convergence we are looking for.

Lemma 1. *Let $(x_n)_n$ be a sequence of nonnegative numbers such that $\frac{x_k}{k} \underset{k}{\to} 0$ and $\frac{\#\{k : \frac{x_k}{k} \geq 1/n\}}{n} \mapsto \bar{x}$, then*

(a) $\lim_{p\to 1^+}(p-1)^{1/p}\left(\sum_{n=1}^{\infty}(\frac{x_n}{n})^p\right)^{1/p} = \bar{x}$.

(b) *If $\frac{x_{k^*}}{k^*}$ is a decreasing rearrangement of the sequence $(\frac{x_k}{k})_k$ then $k \cdot \frac{x_{k^*}}{k^*}$ converges to $\bar{x}$.*

Proof. We denote by $R_n = \{k : \frac{x_k}{k} \geq 1/n\}$ and $N_n = \#\{k : \frac{x_k}{k} \geq 1/n\} = \#R_n$. To prove (a) it is enough to show that

$$\lim_{p\to 1^+}(p-1)\left(\sum_{n=1}^{\infty}(\frac{x_n}{n})^p\right) = \bar{x}.$$

We can write the series $(p-1)(\sum_{n=1}^{\infty}(\frac{x_n}{n})^p)$ in the following way;

$$\begin{aligned}(p-1)\left(\sum_{n=1}^{\infty}(\frac{x_n}{n})^p\right) &= (p-1)\left[\sum_{n\in R_1}(\frac{x_n}{n})^p + \sum_{n\in\mathbb{N}^*\setminus R_1}(\frac{x_n}{n})^p\right] \\ &= A_p + B_p.\end{aligned}$$

As $\lim_{p\to 1} A_p = 0$ we just need to consider $B_p = (p-1)\sum_{n\in\mathbb{N}^*\setminus R_1}(\frac{X_n}{n})^p$. But we have

$$(p-1)\sum_{n=1}^{\infty}\frac{N_{n+1}-N_n}{(n+1)^p} \leq B_p \leq (p-1)\sum_{n=1}^{\infty}\frac{N_{n+1}-N_n}{n^p}.$$

It is then enough to prove that B_p is squeezed into two terms tending to the same limit $\bar{x}$. We will only prove that the term $(p-1)\sum_{n=1}^{\infty}\frac{N_{n+1}-N_n}{n^p}$ converges to $\bar{x}$. The same argument shows the same conclusion for the second term $(p-1)\sum_{n=1}^{\infty}\frac{N_{n+1}-N_n}{(n+1)^p}$.

We have

$$\begin{aligned}(p-1)\sum_{n=1}^{\infty}\frac{N_{n+1}-N_n}{n^p} &= (p-1)\left(-\frac{N_1}{1^p} + \sum_{n=2}^{\infty}\frac{N_n(n^p-(n-1)^p))}{n^p(N-1)^p}\right) \\ &= (p-1)\left[-\frac{N_1}{1^p} + \sum_{n=2}^{\infty}\frac{N_n(1-(\frac{(n-1)}{n})^p))}{(n-1)^p}\right] \\ &\sim (p-1)\left[-\frac{N_1}{1^p} + p\sum_{n=2}^{\infty}\frac{N_n}{n}\cdot\frac{1}{(n-1)^p}\right].\end{aligned}$$

As $\frac{N_n}{n}$ converges to $\bar{x}$ and $\sum_{n=2}^{\infty}\frac{1}{(n-1)^p} \sim \frac{1}{p-1}$ we conclude that

$$\lim_{p\to 1^+}(p-1)\sum_{n=1}^{\infty}\frac{N_{n+1}-N_n}{n^p} = \bar{x}.$$

(b) To obtain the convergence of the sequence $k\frac{x_{k^*}}{k^*}$ to $\bar{x}$ we can observe that

$$\lim_{t\to\infty}\frac{\#\{\ell : \frac{x_\ell}{\ell} \geq \frac{1}{t}\}}{t} = \bar{x}.$$

If we take the increasing sequence $t_k = \frac{k^*}{x_{k^*}}$ where $\frac{x_{k^*}}{k^*}$ is the k^{th} term of the decreasing rearrangement of the sequence $\frac{x_k}{k}$ we can see that

$$\frac{x_{k^*}}{k^*}\cdot\#\{\ell : \frac{x_\ell}{\ell} \geq \frac{x_{k^*}}{k^*}\} = k\cdot\frac{x_{k^*}}{k^*} \qquad \text{converges to } \bar{x}.$$

This ends the proof of this lemma. □

In this part we only consider sequences X_n of iid nonnegative random variables such that $\mathbb{E}(X_1) < \infty$. This assumption can be made in view of the nature of our p series.

Theorem 2. *Let (X_n) be a sequence of iid nonnegative random variables such that $\mathbb{E}(X_1) < \infty$. Then we have*

$$\text{(a)} \qquad \lim_{n\to\infty}\frac{N_n(x)}{n} = \mathbb{E}(X_1) \ \text{a.e.} \ \left(\text{with } N_n(x) = \#\left\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\}\right),$$

$$\text{(b)} \qquad \lim_{p\to 1^+}(p-1)^{1/p}\left(\sum_{n=1}^{\infty}\left(\frac{X_n(x)}{n}\right)^p\right)^{1/p} = \mathbb{E}(X_1), \ \text{a.e.}$$

Proof. By the previous lemma, (b) is an immediate consequence of (a), so we are left with proving (a).

In our proof of Lemma 1 in [A1], we showed that we have

$$\left\|\frac{X_n(x)}{n}\right\|_{1,\infty} < \infty \ \text{ a.e., because } \ \overline{\lim}_{n\to\infty}\frac{\#\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\}}{n} = \mathbb{E}(X_1).$$

We proved this by noting that

$$N_n(x) = \#\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\} = \sum_{n=1}^{\infty}\mathbf{1}\left\{x : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\}.$$

Then we considered

$$V_n(x) = \sum_{n=1}^{\infty}\mathbf{1}\left\{x : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\} - \mu\left\{x : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\}.$$

Kolmogorov's inequality for sums of independent random variables leads to the following inequality for each $\epsilon > 0$.

$$\sum_{n=1}^{\infty}\mu\{\left|\frac{N_{n^2}(x) - \mathbb{E}(N_{n^2})}{n^2}\right| \geq \epsilon\} < \infty.$$

An application of the Borel-Cantelli lemma gave us

$$\overline{\lim}\frac{N_{n^2}(x)}{n^2} = \lim_n \frac{\mathbb{E}(N_{n^2})}{n^2} = \mathbb{E}(X_1).$$

Then a simple interpolation allowed us to claim that

$$\overline{\lim}_n \frac{N_n(x)}{n} = \mathbb{E}(X_1). \tag{8}$$

But also in [A1], Theorem 3 shows that for each p, $1 < p \leq \infty$ we have

$$\lim_{n\to\infty} \frac{\#\{k : \frac{Y_k(x)}{k} \geq 1/n\}}{n} = \mathbb{E}(Y_1) \tag{9}$$

for (Y_n) sequence of iid random variables where $\mathbb{E}(|Y_1|^p) < \infty$ for some $1 < p \leq \infty$.

We take M a positive constant; using (8) and (9) we get

$$\begin{aligned}\mathbb{E}(X_1 \wedge M) &= \lim_n \frac{\#\{k : \frac{X_k(x)\wedge M}{k} \geq 1/n\}}{n}\\ &\leq \underline{\lim}\frac{\#\{k : \frac{X_k(x)}{k} \geq 1/n\}}{n}\\ &= \overline{\lim}\frac{\#\{k : \frac{X_k(x)}{k} \geq 1/n\}}{n}\\ &= \mathbb{E}(X_1).\end{aligned}$$

As $\lim_M \mathbb{E}(X_1 \wedge M) = \mathbb{E}(X_1)$ we have obtained a proof of (a) from which (b) now follows easily. □

2.2. The multidimensional case. The previous situation can be extended to a more general situation. In [A1] we proved the following:

Given H a positive integer and a nonnegative iid sequence $(X_{1,n})_n$ on the probability measure space $(\Omega_1, \mathcal{F}_1, \mu_1)$ satisfying the condition $\mathbb{E}(X_{1,1}) < \infty$, it is possible to find a set of full measure $\widetilde{\Omega}_1$ such that if $x_1 \in \widetilde{\Omega}_1$ the following holds:

For all probability measure spaces $(\Omega_2, \mathcal{F}_2, \mu_2)$ and all nonnegative iid sequences $(X_{2n})_n$ such that $\mathbb{E}(X_{2,1}) < \infty$ it is possible to find a set of full measure $\widetilde{\Omega}_2$ such that if $x_2 \in \widetilde{\Omega}_2$ the following holds:

For all probability measure spaces $(\Omega_H, \mathcal{F}_H, \mu_H)$ and all iid sequences $(X_{H,n})_n$ of nonnegative random variables satisfying $\mathbb{E}(X_{H,1}) < \infty$ we can find a set of full measure $\widetilde{\Omega}_H$ for which if $x_H \in \widetilde{\Omega}_H$ we have

$$\overline{\lim}_n \frac{\#\{k : \frac{\prod_{i=1}^H X_{i,k}(x_i)}{k} \geq \frac{1}{n}\}}{n} = \prod_{i=1}^H \mathbb{E}(X_{i,1}). \tag{10}$$

The difficulty resides in the way those sets of full measure $\widetilde{\Omega}_i$ are obtained; they are independent of the incoming variables $(X_{j,n})$ for $j > i$.

We want to prove that in (10) we actually have convergence to $\prod_{i=1}^H \mathbb{E}(X_{i,1})$. More precisely we have:

Theorem 3. *Given H a positive integer and a nonnegative sequence of iid variables $(X_{1n})_n$ on the probability measure space $(\Omega_1, \mathcal{F}_1, \mu_1)$ satisfying the condition $\mathbb{E}(X_{1,1}) < \infty$, it is possible to find a set of full measure $\widetilde{\Omega}_1$ such that if $x_1 \in \widetilde{\Omega}_1$ the following holds:*

For all probability measure spaces $(\Omega_2, \mathcal{F}_2, \mu_2)$ and all nonnegative iid sequences $(X_{2,n})_n$ such that $\mathbb{E}(X_{2,1}) < \infty$, it is possible to find a set of full measure $\widetilde{\Omega}_2$ such that if $x_2 \in \widetilde{\Omega}_2$ the following holds:

For all probability measure spaces $(\Omega_H, \mathcal{F}_H, \mu, H)$ and all iid sequences $(X_{H,n})_n$ of nonnegative random variables satisfying $\mathbb{E}(X_{H,1}) < \infty$ we can find a set of full measure $\widetilde{\Omega}_H$ for which if $x_H \in \widetilde{\Omega}_H$ we have

$$\lim_n \frac{\#\{k : \frac{\prod_{i=1}^H X_{i,k}(x_i)}{k} \geq \frac{1}{n}\}}{n} = \prod_{i=1}^H \mathbb{E}(X_{i,1}) \tag{11}$$

and

$$\lim_{p\to 1^+} \left((p-1)\left(\sum_{n=1}^\infty \left(\frac{\prod_{i=1}^H X_{i,n}(x_i)}{n}\right)^p\right)^{1/p}\right) = \prod_{i=1}^H \mathbb{E}(X_{i,1}). \tag{12}$$

Proof. As previously we just need to prove (11) to get (12). We use induction to prove (11). The result is true for $H = 1$, as shown in the previous theorem.

Let us assume that the result is true for $H-1$. Hence if $c_k = \prod_{i=1}^{H-1} X_{ik}(x_i)$ where $x_i \in \widetilde{\Omega}_i$ we have

$$\lim_{n\to\infty} \frac{\#\{k : \frac{c_k}{k} \geq \frac{1}{n}\}}{n} = \prod_{i=1}^{H-1} \mathbb{E}(X_{i,1}). \tag{13}$$

The idea of the proof is the same as in Lemma 1 in [A1]. We have for $x_i \in \widetilde{\Omega}_i$, $1 \leq i \leq H-1$, $(X_{H,n})$ a sequence of nonnegative iid random variables and for all $\epsilon > 0$

$$\sum_{n=1}^\infty \mu\left\{x_H : \left|\frac{N_{n^2}(x_H) - \mathbb{E}(N_{n^2})}{n^2}\right| \geq \epsilon\right\} < \infty \tag{14}$$

where

$$N_{n^2}(x_H) = \frac{\#\{k : \frac{c_k X_{H,k}(x_H)}{k} \geq 1/n^2\}}{n^2}.$$

The inequality (14) is obtained by applying Kolmogorov's inequality to the series of independent random variables

$$\sum_{k=1}^\infty \mathbf{1}_{\left\{x_H : \frac{c_k X_{H,k}(x_H)}{k} \geq 1/n\right\}} - \mu\left\{x_H : \frac{c_k X_{H,k}(x_H)}{k} \geq 1/n\right\}.$$

The Borel-Cantelli lemma applied to (14) gives us

$$\lim_{n\to\infty} \frac{N_{n^2}(x_H) - \mathbb{E}(N_{n^2})}{n^2} = 0 \qquad \text{a.e. } (x_H).$$

As $\lim_{n\to\infty} \frac{\mathbb{E}(N_{n^2})}{n^2} = \prod_{i=1}^{H} \mathbb{E}(X_{i,1})$ we have

$$\lim_{n\to\infty} \frac{N_{n^2}(x_H)}{n^2} = \prod_{i=1}^{H} \mathbb{E}(X_{i,1}) \qquad \text{a.e. } (x_H).$$

The monotonicity of N_n gives us for $p_n^2 \le n \le (p_{n+1})^2$

$$\frac{N_{p_n^2}(x_H)}{p_n^2} \le \frac{N_n(x_H)}{p_n^2} \le \frac{N_{(p_{n+1})^2}(x_H)}{p_n^2} = \frac{N_{(p_{n+1})^2}(x_H)}{(p_{n+1})^2} \cdot \frac{(p_{n+1})^2}{(p_n)^2}.$$

This last chain of inequalities implies that

$$\lim_{n\to\infty} \frac{N_n(x_H)}{n} = \lim_{n\to\infty} \frac{N_{p_n^2}(x_H)}{p_n^2} = \prod_{i=1}^{H} \mathbb{E}(X_{i,1}) \qquad \text{as } \frac{p_n^2}{n} \to 1.$$

□

3. Convergence of the p-series for ergodic stationary sequences

In this part the sequence X_n will be given by an ergodic dynamical system $(X, \mathcal{F}, \mu, T)$ on a probability measure space $(X, \mathcal{F}, \mu)$. The sequence is defined by the relation $X_n(x) = f(T^n x)$ where f is a nonnegative integrable function.

Proposition 4. *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic dynamical system and f a nonnegative integrable function. We have*

$$\lim_{n\to\infty} \left\| \frac{N_n(f)}{n} - \int f d\mu \right\|_1 = 0, \quad \textit{where}$$
$$\frac{N_n(f)(x)}{n} = \frac{\#\{k : \frac{f(T^k x)}{k} \ge 1/n\}}{n}$$

Proof. We know that $\lim_{n\to\infty} \frac{N_n(f)}{n} = \int f d\mu$ a.e. for $f \in L_+^p$ for some p, $1 < p \le \infty$ (see Theorem 3 in [A1]). The difficulty at this level comes from the nature of the function of f, $N_n(f)$; the map N_n is not linear nor positively homogeneous. But we have the following properties:

(A) $\|\frac{N_n(f)}{n}\|_\infty \le \|f\|_\infty$,
(B) If f, g are nonnegative functions with disjoint support then we have $\frac{N_n(f+g)}{n} = \frac{N_n(f)}{n} + \frac{N_n(g)}{n}$ for all $n \ge 1$.
(C) For all $f \ge 0$ integrable functions we have $\|\frac{N_n(f)}{n}\|_1 \le \|f\|_1$.

(A) and (B) are easy to check.

To establish (C) we take $f \in L^1$ for which we can find for each ϵ nonnegative numbers $(\alpha_i)_i$ and sets $(A_i)_i$ such that $f \leq \sum \alpha_i \ \mathbf{1}_{A_i}, \ A_i \cap A_j = \phi$ if $i \neq j$ and $\int \sum \alpha_i \mathbf{1}_{A_i} d\mu \leq (1+\epsilon) \int f d\mu$. We have

$$\frac{N_n(f)}{n} \leq \frac{N_n(\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{A_i})}{n} \qquad \text{by monotonicity.}$$

Thus

$$\begin{aligned} \left\| \frac{N_n(f)}{n} \right\|_1 &\leq \left\| \frac{N_n(\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{A_i})}{n} \right\|_1 \\ &= \left\| \sum_{i=1}^{\infty} \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} \right\|_1 \qquad \text{by (B)} \\ &= \sum_{i=1}^{\infty} \left\| \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} \right\|_1 . \end{aligned}$$

As

$$\begin{aligned} \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} &= \frac{\#\{k : \frac{\mathbf{1}_{A_i}(t^k x)}{k} \geq \frac{1}{n\alpha_i}\}}{n} \\ &= \frac{\sum_{k=1}^{[n\alpha_i]} \mathbf{1}_{A_i}(T^k x)}{n} \qquad \text{we have} \end{aligned}$$

$$\left\| \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} \right\|_1 = \sum_{k=1}^{[n\alpha_i]} \frac{\mu(A_i)}{n} \leq \frac{(n\alpha_i)\mu(A_i)}{n} = \alpha_i \mu(A_i) \, .$$

So

$$\left\| \frac{N_n(f)}{n} \right\| \leq \sum_{i=1}^{\infty} \alpha_i \mu(A_i) \leq (1+\epsilon) \int f d\mu.$$

As ϵ is arbitrary we have reached a proof of (C).

We are now in a position to prove Proposition 4.

For each positive real number M we can write $f = f \wedge M + g_M$ with $f \wedge M$ and g_M nonnegative functions with disjoint support.

We have

$$\frac{N_n(f)}{n} - \int f d\mu = \frac{N_n(f \wedge M)}{n} - \int f \wedge M d\mu + \frac{N_n(g_M)}{n} - \int g_M d\mu.$$

Hence

$$\begin{aligned} \overline{\lim_n} \left\| \frac{N_n(f)}{n} - \int f d\mu \right\|_1 \leq \overline{\lim_n} &\left\| \frac{N_n(f \wedge M)}{n} - \int (f \wedge M) d\mu \right\|_1 \\ &+ \overline{\lim_n} \left\| \frac{N_n(g_M)}{n} \right\|_1 + \int g_M d\mu. \end{aligned}$$

By the theorem mentioned at the beginning of this proof, associating the a.e. convergence of $\frac{N_n(f)(x)}{n}$ to $\int f d\mu$ for functions in L^p for some p, we conclude that

$$\overline{\lim_n} \left\| \frac{N_n(f \wedge M)}{n} - \int f \wedge M d\mu \right\|_1 = 0.$$

Hence

$$\overline{\lim_n} \left\| \frac{N_n(f)}{n} - \int f d\mu \right\|_1 \leq 2 \int g_M d\mu, \qquad \text{by (C).}$$

As $\int g_M d\mu \underset{M}{\longrightarrow} 0$, the proof of this proposition is complete. □

Theorem 5. *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic dynamical system and $f \in L \log L,\ f \geq 0$. Then we have*

(a) $$\left\| \left\| \left(\frac{f(T^k x)}{k} \right) \right\|_{1,\infty} \right\|_1 = \left\| \sup_n n \cdot \frac{f(T^{n^*} x)}{n^*} \right\|_1 < \infty$$

where $\frac{f(T^{n^*} x)}{n^*}$ is for μ a.e. x a decreasing rearrangement of the sequence $\frac{f(T^n x)}{n}$.

(b) $$\lim_{n \to \infty} \frac{N_n(f)(x)}{n} = \int f d\mu,\ \mu \text{ a.e.}$$

(c) $$\lim_{p \to 1+} \left((p-1) \sum_{n=1}^{\infty} \left(\frac{f(T^n x}{n} \right)^p \right)^{1/p} = \int f d\mu, \mu \text{ a.e.}$$

Proof. First we can make the following observations:
For all measurable sets A we have

$$\begin{aligned} \left\| \left(\frac{\mathbf{1}_A(T^k x)}{k} \right) \right\|_{1,\infty} &= \sup_{t>0} \frac{\#\{k : \frac{\mathbf{1}_A(T^k x)}{k} \geq 1/t\}}{t} = \sup_n \frac{\#\{k : \frac{\mathbf{1}_A(T^k x)}{k} \geq 1/n\}}{n} \\ &= \sup_n \frac{N_n(\mathbf{1}_A)(x)}{n} \\ &= N^*(\mathbf{1}_A)(x). \end{aligned} \tag{15}$$

Because of the maximal inequality for the ergodic averages we have

$$\mu\{x : N^*(\mathbf{1}_A)(x) > \lambda\} \leq \frac{1}{\lambda} \cdot \mu(A) \qquad \text{for all } \lambda > 0. \tag{16}$$

(Note that $N^*(\mathbf{1}_A)(x) \leq 1$, hence for all $p \geq 1$ we also have

$$\mu\{x : N^*(\mathbf{1}_A)(x) > \lambda\} \leq \frac{1}{\lambda^p} \cdot \mu(A)). \tag{17}$$

For all positive real numbers y we have:

$$y^{\frac{i}{i+1}} = y^{\frac{i}{i+1}} \cdot \frac{(i+1)^{1/i+1}}{(i+1)^{1/i+1}} \leq \frac{y(i+1)^{1/i}}{(i+1)} i + \frac{1}{(i+1)^2} \tag{18}$$

(apply the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, for $a = y^{i/i+1} \cdot (i+1)^{1/i+1}$, $b = \frac{1}{(i+1)^{1/i+1}}$, $p = \frac{i+1}{i}$ and $q = \frac{p}{p-1} = i+1$).

We proceed now with the proof of Theorem 5 (a).

We take $f \in L \log L$ and denote by A_i the set

$$A_i = \{2^i \leq f < 2^{i+1}\}.$$

We have

$$\begin{aligned} N^*(f) \leq N^*\left(\sum_{i=1}^{\infty} 2^{i+1}\mathbf{1}_A)\right) &\leq \sum_{i=1}^{\infty} N^*(2^{i+1}\mathbf{1}_A)) \\ &= 2\sum_{i=1}^{\infty} 2^i \cdot N^*(\mathbf{1}_A). \end{aligned}$$

By taking the integral with respect to the measure μ we get

$$\|N^*(f)\|_1 \leq 2\sum_{i=1}^{\infty} 2^i \|N^*(\mathbf{1}_{A_i})\|_1$$

Using (17) we get

$$\begin{aligned} \|N^*(\mathbf{1}_{A_i})\|_1 &\leq \frac{p}{(p-1)} \sup_{t>0}[t \cdot \mu\{x : n^*(\mathbf{1}_{A_i})(x) > t\}] \\ &\leq \frac{p}{(p-1)} \cdot (\mu(A_i))^{1/p} \qquad \text{for all } p,\ 1 \leq p < \infty. \end{aligned}$$

((17) is combined with the inequality $\|g\|_{L^1} \leq \frac{p}{(p-1)} \sup_{t>0}[t\mu\{x : |g(x)| > t\}^{1/p}]$.) Going back to the evaluation of $\|N^*(f)\|_1$ we get

$$\begin{aligned} \|N^*(f)\|_1 &\leq 2\sum_{i=1}^{\infty} 2^i \frac{(i+1/i)}{1/i} (\mu(A_i))^{1/i+1} \\ &= 2\sum_{i=1}^{\infty} 2^i (i+1)(\mu(A_i))^{i/i+1} \\ &= 2\sum_{i=1}^{\infty} ((2^i(i+1))^{i+1/i}\mu(A_i))^{i/i+1}. \end{aligned}$$

Applying (18) to each term $((2^i(i+1))^{i+1/i}\mu(A_i))^{i/i+1}$ we get

$$\begin{aligned} \|N^*(f)\|_1 &\leq 2\sum_{i=1}^{\infty}[(2^i(i+1))^{i+1/i} \cdot (\mu(A_i))\frac{(i+1)^{1/i}}{i+1} i + \frac{1}{(i+1)^2}] \\ &\leq 4\sum_{i=1}^{\infty}[2^i i\mu(A_i) \cdot (1+i)^{2/i} + \frac{1}{(i+1)^2}] \\ &\leq 12 \cdot \sum_{i=1}^{\infty}[2^i i\mu(A_i) + \frac{1}{(i+1)^2}] \\ &\leq \frac{12}{\ln 2}[\int f \log f d\mu + 1]. \end{aligned}$$

Thus we have proved the following inequality

$$\|N^*(f)\|_1 \le \frac{12}{\ln 2}[\int f \log f d\mu + 1] \qquad \text{for all } f \ge 0,\ f \in L\log L. \tag{19}$$

This clearly ends the proof of Theorem 5 (a).

It remains to show (b). Our goal is to prove that for $f \ge 0,\ f \in L\log L$

$$\underline{\lim_n} N^*(f - f\wedge n) = 0 \text{ a.e.} \tag{20}$$

Using (19) we have for all $t > 0$,

$$\|N^*(t(f - f\wedge n))\|_1 \le \frac{12}{\ln 2}[\int (t(f - f\wedge n))\log(t(f - f\wedge n))d\mu + 1]$$
$$\text{for all } f \ge 0,\ f \in L\log L.$$

This last inequality gives us

$$\|N^*(f - f\wedge n)\|_1 \le \frac{12}{\ln 2}[\int (f - f\wedge n)\log(t(f - f\wedge n)]d\mu + \frac{1}{t}].$$

At the expense of taking a subsequence, we derive from it

$$\lim_k \|N^*(f - f\wedge n_k)\|_1 \le \frac{12}{\ln 2}\cdot\frac{1}{t}.$$

Then we easily get $\underline{\lim_n} N^*(f - f\wedge n) = 0$ a.e. This proves (20).

As $\lim_n \frac{N_k(f\wedge n)}{k} = \int f\wedge n d\mu$, because $f\wedge n$ is clearly bounded we have

$$\frac{N_k(f\wedge n)}{k} \le \frac{N_k(f)}{k} = \frac{N_k(f\wedge n)}{k} + \frac{N_k(f - f\wedge n)}{k}$$

and after taking the limits we obtain

$$\int f\wedge n d\mu \le \underline{\lim_k}\frac{N_k(f)}{k} \le \overline{\lim}\frac{N_k(f)}{k} \le \overline{\lim}\frac{N_k(f\wedge n)}{k} + N^*[f - f\wedge n]$$
$$= \int f\wedge n d\mu + N^*[f - f\wedge n].$$

Finally, by taking the lim inf with respect to n we can conclude that

$$\lim_k \frac{N_k(f)}{k} = \int f d\mu \text{ a.e.}$$

This proves Theorem 5 (b). Theorem 5 (c) now follows easily from Lemma 1. This ends the proof of Theorem 5. □

Corollary 6. *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic dynamical system and $f \in L \log L,\ f \geq 0$. Then there exists an absolute constant C such that*

$$\left\| \sup_{1<p<\infty} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} \right\|_1 \leq C[\int f \log f d\mu + 1]$$

Proof. By Theorem 5 we know that

$$\left\| \sup_n n \cdot \frac{f(T^{n^*} x)}{n^*} \right\|_1 < \infty.$$

As for μ a.e. x , for each p, we have

$$(p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} \leq \left(\sup_n n \cdot \frac{f(T^{n^*} x)}{n^*} \right) \cdot \left((p-1) \sum_{n=1}^{\infty} 1/n^p \right)^{1/p},$$

the corollary follows easily. □

Remark.

1) One can see that the limit when p tends to ∞ of the p-Series is equals to $\sup_n \frac{f(T^n x)}{n}$. This is the reason why we only focus on the existence of the limit when p tends to 1+.
2) We proved in [A2] that if $N^*(f)(x)$ is a.e finite for all functions $f \in L^1_+$ then $M_2^*(f)(x)$ is also a.e finite for all functions $f \in L^1$.
3) The results obtained in this note can be extended to increasing sequences of integers $(p_n)_n$. The corresponding maximal function to consider is simply

$$\| \left(\frac{f(T^{p_n})(x)}{n} \right) \|_{1,\infty}.$$

To illustrate this we have the following Proposition.

Proposition 7. *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic dynamical system , p a fixed positive real number $1 < p < \infty$ and $(p_n)_n$ an increasing sequence of positive integers. Consider the following statements*

(a) $$\left\| \left(\frac{f(T^{p_n} x)}{n} \right) \right\|_{1,\infty} < \infty \text{ a.e. for all } f \in L^p_+.$$

(b) $$\sup_k k^{p-1} \sum_{n=k}^{\infty} \left(\frac{f(T^{p_n} x)}{n} \right)^p < \infty \text{ a.e. for all } f \in L^p_+.$$

(c) $$\sup_N \frac{1}{N} \sum_{n=1}^{N} f(T^{p_n} x) < \infty \text{ a.e. for all } f \in L^p_+.$$

(d) $$\sup_N \frac{1}{N} \sum_{n=1}^{N} f(T^{p_n} x) < \infty \text{ a.e. for all } f \in L(p,1).$$

Then we have the following implications: (a) *implies* (d), (b) *is equivalent to* (c), (b) *implies* (a) *and* (c) *implies* (d).

Proof. The implications (c) implies (b) and (b) implies (a) can be proved the same way we did in [A1] for the usual Cesaro averages. (See the proof of Theorem 3 part b) in [A1]). The implication (c) implies (d) is a direct consequence of the structure of $L(p,q)$ spaces as shown in [SW].

It remains to prove the implications (a) implies (d) and (b) implies (c). For (a) implies (d), we can notice that (a) implies the existence of a finite constant C_p such that for all $f \in L_+^p$

$$\mu\{x : \left\| \left(\frac{f(T^{p_n} x)}{n} \right) \right\|_{1,\infty} > \lambda\} \leq \frac{C_p}{\lambda^p} \int |f|^p d\mu \qquad \text{for all} \quad \lambda > 0. \tag{21}$$

In the particular case of $f = \mathbf{1}_A$, (21) will give us the following

$$\mu\{x : \sup_N \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_A(T^{p_n} x) > \lambda\} \leq \frac{C_p}{\lambda^p} \mu(A) \qquad \text{for all} \quad \lambda > 0 \tag{22}$$

because

$$\left\| \left(\frac{\mathbf{1}_A(T^{p_n} x)}{n} \right) \right\|_{1,\infty} = \sup_N \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_A(T^{p_n} x).$$

As this inequality is valid for all measurable sets A, we can conclude that the maximal operator

$$M^*(f)(x) = \sup_N \frac{1}{N} \sum_{n=1}^{N} f(T^{p_n} x)$$

is of restricted weak type (p,p) (see [SW]). In other words, the maximal operator M^* maps the characteristic function of any measurable set A from $L(p,1)$ into $L(p,\infty)$. It is shown in [SW] that the nature of the maximal operator M^* and the existence of an equivalent norm on $L(p,\infty)$, making it a Banach space, M^* maps continuously all functions $f \in L(p,1)$ into $L(p,\infty)$. This means the existence of a finite constant C_p such that for all $f \in L(p,1)$,

$$\mu\{x : \sup_N \frac{1}{N} \sum_{n=1}^{N} f(T^{p_n} x) > \lambda\} \leq \frac{C_p}{\lambda^p} \|f\|_{p,1}{}^p \qquad \text{for all} \quad \lambda > 0. \tag{23}$$

From this we can clearly derive (d).

The implication (b) implies (c) can be obtained by summation. For $f \in L_+^p$ let us denote by C_x the finite constant which dominates the sup on k. Then for each k, we have

$$k^{p-1} \sum_{n=k}^{2k} \left(\frac{f(T^{p_n} x)}{2k} \right)^p < C_x.$$

This implies the inequality

$$\sup_k \frac{1}{k} \sum_{n=k}^{2k} f(T^{p_n} x)^p < 2^p C_x.$$

From this we can derive by convexity the uniform boundedness of the averages $\frac{1}{k} \sum_{n=k}^{2k} f(T^{p_n} x)$. This property allows us to obtain (c) without difficulty. $\square$

Remark. It would be interesting to know if (a) implies (c) for any increasing sequence p_n and any p, $1 \leq p < \infty$. For $p = 1$, $p_n = n$ we already mentioned that (a) implies (c) (see [A2]).

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