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Harnack Inequalities For Curvature Flows Depending On Mean Curvature

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Abstract. We prove Harnack inequalities for parabolic flows of compact orientable hypersurfaces in $\mathbb{R}^{n+1}$, where the normal velocity is given by a smooth function f depending only on the mean curvature. We use these estimates to prove longtime existence of solutions in some highly nonlinear cases. In addition we prove that compact selfsimilar solutions with constant mean curvature must be spheres and that compact selfsimilar solutions with nonconstant mean curvature can only occur in the case, where $f = A\alpha x^\alpha$ with two constants A and α.

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1. Introduction

Assume that M^n is a compact orientable surface, smoothly immersed into $\mathbb{R}^{n+1}$ by a smooth family of diffeomorphisms $F_t : M^n \to \mathbb{R}^{n+1}$ that satisfy the PDE

$$\frac{\partial}{\partial t} F = -f\nu \ ,$$

where ν denotes the outward pointing unit normal and f is a smooth function depending only on the mean curvature H of the immersed surface, e.g. for $f = H$ we get the well-known mean curvature flow (MCF) and for $f = -\frac{1}{H}$ we obtain the inverse mean curvature flow. Hamilton [4] proved a beautiful Harnack inequality for the MCF. In [1] Harnack inequalities were derived for convex hypersurfaces in cases where f may depend on the full second fundamental form. The case $f = -\frac{1}{H}$

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has not been studied yet nor are there results for arbitrary functions f depending only on H. The aim of this paper is to address the following questions:

I. What conditions on f guarantee that the flow becomes parabolic so that we have shorttime existence of a solution?
II. For which f do we get nice Harnack inequalities?
III. Since selfsimilar solutions play an important role in the Harnack inequality for the MCF, we ask: For which f do selfsimilar solutions exist and can we say something about the nature of these solutions?

An interesting case is the inverse mean curvature flow $f = -\frac{1}{H}$ since it is important in General Relativity [6]. There is some hope that one can generalize our results to other target manifolds N^{n+1}.

Throughout this paper we will use the standard terminology, i.e., $\langle \cdot, \cdot \rangle$ denotes the euclidean inner product, $g_{ij}dx^i \otimes dx^j$ is the induced metric on M^n, x^i coordinates for M^n and $h_{ij}dx^i \otimes dx^j = \langle \nabla_i F, \nabla_j \nu \rangle dx^i \otimes dx^j$ denotes the second fundamental form, ∇ the covariant derivative with respect to g_{ij}. Double latin indices are summed from 1 to n and we set

$$|A|^2 := h_{ij}h^{ij} \ , \ C := h_{ik}h_l{}^k h^{il} \ .$$

In this paper we will always assume that $f : \Omega \to \mathbb{R}$ is a smooth function defined on an open subset in $\mathbb{R}$ and we define

Definition. Let $F : M^n \to \mathbb{R}^{n+1}$ be an immersion. F is called admissible, if $H(x) \in \Omega, \ \forall x \in M^n$.

The answer to question I is then given by

Proposition I. *Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be an admissible smooth immersion of a compact orientable surface M^n and assume that $f' : \Omega \to \mathbb{R}$ is strictly positive. Then the PDE*

$$(\star) \qquad \begin{aligned} \frac{\partial}{\partial t}F &= -f\nu \\ F(x,0) &= F_0(x), \ \forall x \in M^n \end{aligned}$$

has a smooth admissible solution on a maximal time interval $[0,T), \ T > 0$.

Proof. This follows from the fact that the linearization of $(\star)$ differs from the linearization for the mean curvature flow only by a factor f' which by assumption is strictly positive. Therefore $(\star)$ is a (nonlinear) parabolic equation and the compactness of M^n and the theory for parabolic equations imply shorttime existence. $\square$

In view of Proposition I we will always assume that $f' > 0$. Then the main theorem can be stated as follows

Theorem 1. *Assume that $F_0 : M^n \to \mathbb{R}^{n+1}$ is an admissible smooth and convex immersion of an orientable compact M^n and that $f : \Omega \to \mathbb{R}$ is a smooth function such that for all $x \in \Omega$ we have*

$$f' > 0, \ \frac{f''}{f'}x^2 \geq ax, \ \big(\frac{f''}{f'}x\big)' \leq 0, \ ff''x + ff' - (f')^2 x \geq 0$$

where $a \in \mathbb{R}$ is a constant. Then we can find a small positive constant d such that

$$\frac{\partial}{\partial t} f + 2\langle \nabla f, V\rangle + h_{ij}V^iV^j + cf'H \geq 0$$

holds for all tangent vectors V as long as $t < T$, $d + (a+2)t > 0$ and M_t stays convex, where we have set $c(t) := \frac{1}{d+(a+2)t}$

Remark. We do have to make these assumptions on f to avoid negative terms in the evolution equation for the basic Harnack expression Z (see below). $f = \alpha x^\alpha$ satisfies the assumptions in Theorem 1 on $\Omega = (0,\infty)$ with $a = \alpha - 1$. The following Propositions show that almost all functions satisfying the assumptions in Theorem 1 are of this form.

Proposition IIa. *Assume that $f : (0,a) \to \mathbb{R}$, $0 < a \leq \infty$ is a smooth function that smoothly extends to $x = 0$ and satisfies all assumptions in Theorem 1. Let us set $i_f := \min\{l \geq 0 | f^{(l)}(0) \neq 0\} \leq \infty$. If $0 < i_f < \infty$ then $f = Ai_f x^{i_f}$ with a positive constant A.*

Proof. Since $f(0) = 0$ and $f'(x) > 0$, $\forall x \in \Omega = (0,a)$ we observe that $f(x) > 0$, $\forall x \in \Omega$. By de l'Hospital's rule we obtain

$$\lim_{x\to 0} \frac{f'}{f} x = i_f$$

and

$$\lim_{x\to 0} \frac{f''}{f'} x = i_f - 1$$

and then $\left(\frac{f''}{f'}x\right)' \leq 0$ and $f' \geq 0$ imply

$$f''x \leq (i_f - 1)f' \ , \ \forall x \tag{1.1}$$

Since $f_{|\Omega} > 0$ we have by assumption

$$ff''x + ff' - (f')^2 x = f^2\left(\frac{f'}{f}x\right)' \geq 0$$

and then also

$$f'x \geq i_f f \ , \ \forall x \tag{1.2}$$

But (1.1) and (1.2) imply

$$0 \leq ff''x + ff' - (f')^2 x \leq (i_f - 1)ff' + ff' - i_f ff' = 0$$

and therefore

$$\left(\frac{f'}{f}x\right)' = 0$$

which implies that

$$\frac{f'}{f}x = i_f$$

and after integration

$$f = Ai_f x^{i_f}$$

with a positive constant A (since $f' > 0$). ☐

Proposition IIb. *Assume that $f:(a,\infty)\to\mathbb{R}$, $0<a<\infty$ is a smooth function that satisfies all assumptions in Theorem 1 and that $g(x):=-f(\frac{1}{x})$ smoothly extends to $x=0$. If $0<i_g<\infty$ then $f=-Ai_g x^{-i_g}$ with a positive constant A.*

Proof. If we set $y=\frac{1}{x}$ and $g(y)=-f(x)$ then the assumptions for f and x translate into the same assumptions for g and y and as above we can derive the desired result. □

Remark. $f(x)=\sqrt{x}$ does not satisfy the assumptions in Proposition IIa since it does not extend smoothly to $x=0$. $f(x)=x+1$ satisfies all assumptions in Theorem 1. $f(x)=\ln x$ satisfies almost all assumptions in Theorem 1. The only condition which is violated is that $ff''x+ff'-(f')^2x=-\frac{1}{x}<0$ on $\Omega=(0,\infty)$.

2. Some relations and evolution equations

By assumption we have

$$\frac{\partial}{\partial t}F=-f\nu \tag{2.1}$$

In [7] we formally derived the evolution equations for various geometric objects on M^n, these are:

$$\frac{\partial}{\partial t}g_{ij}=-2fh_{ij} \tag{2.2}$$

$$\frac{\partial}{\partial t}d\mu=-Hfd\mu \tag{2.3}$$

($d\mu$ denotes the volume form)

$$\frac{\partial}{\partial t}\nu=\nabla f \tag{2.4}$$

$$\frac{\partial}{\partial t}h_{ij}=\nabla_i\nabla_j f-fh_i^{\ l}h_{lj} \tag{2.5}$$

$$\frac{\partial}{\partial t}H=\Delta f+f|A|^2 \tag{2.6}$$

$$\frac{\partial}{\partial t}|A|^2=2\langle h_{ij},\nabla_i\nabla_j f\rangle+2fC \tag{2.7}$$

In addition we have the well-known Gauß-Weingarten-Codazzi-Mainardi equations

$$\nabla_i\nabla_j F=-h_{ij}\nu \tag{2.8}$$

$$\Delta F=g^{ij}\nabla_i\nabla_j F=-H\nu \tag{2.9}$$

$$\nabla_i h_{jk}=\nabla_j h_{ik} \tag{2.10}$$

$$\nabla_i\nu=h_i^{\ l}\nabla_l F \tag{2.11}$$

$$\nabla_i\nabla_j\nu=\nabla^l h_{ij}\nabla_l F-h_i^{\ l}h_{lj}\nu \tag{2.12}$$

$$\Delta\nu=\nabla H-|A|^2\nu \tag{2.13}$$

Note that in these equations we assume that F, ν are sets of $n+1$ functions on M^n.

We also have the Simons identity

$$\nabla_i \nabla_j H = \Delta h_{ij} - H h_i{}^l h_{lj} + |A|^2 h_{ij} \tag{2.14}$$

3. Homothetic solutions

A homothetic solution F_t is a family of diffeomorphisms such that the surfaces given by the rescaled diffeomorphisms $\widetilde{F}_t := \Psi F$ are stationary in $\mathbb{R}^{n+1}$, where Ψ denotes a function depending only on time t. The assumption that $\widetilde{F}_t$ represents a stationary surface means that the normal velocity must be zero. So we have

$$0 = \langle \frac{\partial}{\partial t} \widetilde{F}, \widetilde{\nu} \rangle = \frac{\partial}{\partial t} \Psi \langle F, \nu \rangle - f \Psi$$

Let us define $c := -\frac{\partial}{\partial t} \ln \Psi$. Then we have shown that for a homothetic solution of $(\star)$ we have

$$f = -c \langle F, \nu \rangle \tag{H.1}$$

Taking covariant derivatives of f and using (2.11), (2.10), (2.8) we obtain with $\widetilde{V}_i := c \langle F, \nabla_i F \rangle$

$$\nabla_i f = -h_i{}^l \widetilde{V}_l \tag{H.2}$$

$$\nabla_i \widetilde{V}_j = c g_{ij} + f h_{ij} \tag{H.3}$$

$$\nabla_i \nabla_j f = -\nabla^l h_{ij} \widetilde{V}_l - c h_{ij} - f h_i{}^l h_{lj} \tag{H.4}$$

$$\Delta f + f |A|^2 = -\langle \nabla H, \widetilde{V} \rangle - cH \tag{H.5}$$

$$\Delta \widetilde{V}_j = f \nabla_j H + h_{ji} \nabla^i f \tag{H.6}$$

4. The Harnack inequality

In the sequel we have to calculate many evolution equations. To avoid too complicated formulas it is most convenient to work with coordinates associated to a moving frame. We use similar moving frame coordinates as in [4]. Here the moving frame $\{E_a\}_{a=1,\ldots,n}$ evolves according to

$$\frac{\partial}{\partial t} E_a^i = f h^i{}_j E_a^j \tag{4.1}$$

and we denote the coordinates of a vector V with respect to the moving frame by y_a^i. The following calculations closely follow the procedure in [4]. As in this paper we have

$$\nabla_a^b := y_a^i \frac{\partial}{\partial y_i^b} \tag{4.2}$$

$$D_a := y_a^i \left(\frac{\partial}{\partial x^i} - \Gamma_{ij}^k y_b^j \frac{\partial}{\partial y_b^k} \right) \tag{4.3}$$

$$\delta_{ab} := g_{ac} \nabla_b^c - g_{bc} \nabla_a^c \tag{4.4}$$

In addition we define

$$D_t := \frac{\partial}{\partial t} + f h_a{}^b \nabla_b^a \tag{4.5}$$

Then straightforward computations give the commutator relations

$$[D_a, D_b] = R^d{}_{eba} \nabla_d^e \tag{4.6}$$

$$[\nabla_b^a, D^c] = -I_b^c D^a \ ; \ [\nabla_b^a, D_c] = I_c^a D_b \tag{4.7}$$

$$[\delta_{bc}, D^a] = I_b^a D_c - I_c^a D_b \tag{4.8}$$

$$[\frac{\partial}{\partial t}, D_a] = \big(D_a(f h_b{}^d) + D_b(f h_a{}^d) - D^d(f h_{ab})\big)\nabla_d^b \tag{4.9}$$

$$[D_t, D_a] = D^b(f h_a{}^c)\delta_{bc} + f h_a{}^b D_b \tag{4.10}$$

$$[\Delta, D_a] = D^b(R^d{}_{lab}\nabla_d^l) + R^d{}_{lab}\nabla_d^l D^b \tag{4.11}$$

$$\begin{aligned}[D_t - f'\Delta, D_a] &= f' R_a{}^{bcd} D_b \delta_{cd} + f' D_a h^{bc} h_b{}^d \delta_{cd} + \frac{f''}{f'} D_a f \Delta \\ &\quad + (f - f'H) h_a{}^l D_l + f' h_a{}^n h_n{}^l D_l \end{aligned} \tag{4.12}$$

(4.13)

$$[D_t, \Delta] = (\frac{f}{f'} - H) D_n f D^n + 2 f h_a{}^b D_b D^a + 2 h_{an} D^a f D^n + D^a(D^b(f h_a{}^c)\delta_{bc})$$

In the moving frame the evolution equations reduce to

$$D_t g_{ab} = 0 \ ; \ g_{ab} = I_{ab} \tag{4.14}$$

$$D_t h_{ab} = D_a D_b f + f h_a{}^n h_{nb} \tag{4.15}$$

$$D_t f = f'(\Delta f + f|A|^2) \tag{4.16}$$

Let us now define the following tensors, where we assume that V_a is an arbitrary tangent vector on M^n and c a smooth function to be determined later and only depending on time t.

$$\begin{aligned} X_a &:= D_a f + h_a{}^l V_l \\ Y_{ab} &:= D_a V_b - f h_{ab} - c g_{ab} \\ Z &:= D_t f + 2\langle Df, V\rangle + h_{ab} V^a V^b + c f' H \\ W_{ab} &:= D_t h_{ab} + D^l h_{ab} V_l + c h_{ab} \\ W &:= D_t f + \langle Df, V\rangle + c f' H \end{aligned}$$

By (H.1)–(H.6) these tensors vanish on homothetic solutions if we take $V_a = \widetilde{V}_a$ and the induced c. On the other hand we have

$$\begin{aligned} D_t \widetilde{V}_a &= \frac{\partial}{\partial t}(c\langle F, D_a F\rangle) + f h_c{}^d \nabla_d^c \widetilde{V}_a \\ &= \frac{\partial}{\partial t}(lnc)\widetilde{V}_a - c\langle F, D_a(f\nu)\rangle + f h_a{}^d \widetilde{V}_d \\ &= \frac{\partial}{\partial t}(lnc)\widetilde{V}_a - c\langle F, \nu\rangle D_a f \end{aligned}$$

and therefore

$$(4.17)\qquad (D_t - f'\Delta)\widetilde{V}_a = \frac{\partial}{\partial t}(\ln c)\widetilde{V}_a + f' h_a{}^b h_b{}^c \widetilde{V}_c$$

and on a homothetic solution

$$(\mathrm{H}.7)\qquad (D_t - f'\Delta)\widetilde{V}_a = \frac{\partial}{\partial t}(\ln c)\widetilde{V}_a - f' h_{ab} D^b f$$

In view of (H.7) we define

$$U_a := (D_t - f'\Delta)V_a - \frac{\partial}{\partial t}(\ln c)V_a + f' h_{ab} D^b f$$

which also vanishes on a homothetic solution.

We want to calculate $(D_t - f'\Delta)Z$. We do this in several steps. Using (4.12) and (4.16) we obtain

$$(D_t - f'\Delta)D_a f = \frac{f''}{f'} D_a f \Delta f + (f - f'H)h_a{}^b D_b f + f' h_a{}^n h_n{}^l D_l f + D_a(ff'|A|^2)$$

and therefore

(4.18)

$$\begin{aligned}(D_t - f'\Delta)D_a f =& f'|A|^2 D_a f + 2ff'h^{bc} D_a h_{bc} + f' h_a{}^n h_n{}^l D_l f \\ &+ (f - f'H)h_a{}^b D_b f + \frac{f''}{(f')^2} D_a f D_t f\end{aligned}$$

Further, we use Simons identity to rewrite (4.15)

$$\begin{aligned}D_t h_{ab} &= D_a D_b f + f h_a{}^n h_{nb} \\ &= f'(\Delta h_{ab} - H h_a{}^n h_{nb} + |A|^2 h_{ab}) + \frac{f''}{(f')^2} D_a f D_b f + f h_a{}^n h_{nb}\end{aligned}$$

This gives

$$(4.19)\qquad (D_t - f'\Delta)h_{ab} = f'|A|^2 h_{ab} + (f - f'H)h_a{}^n h_{nb} + \frac{f''}{(f')^2} D_a f D_b f$$

(4.18), (4.19) and the definition of U give

$$\begin{aligned}(D_t - f'\Delta)X_a =\;& f'|A|^2 D_a f + 2ff'h^{bc} D_a h_{bc} + f' h_a{}^n h_n{}^l D_l f + (f - f'H)h_a{}^b D_b f \\ &+ \frac{f''}{(f')^2} D_a f D_t f + V^b(f'|A|^2 h_{ab} + (f - f'H)h_a{}^n h_{nb} + \frac{f''}{(f')^2} D_a f D_b f) \\ &+ h_a{}^b(U_b + \frac{\partial}{\partial t}(\ln c)V_b - f' h_{bc} D^c f) - 2f' D^c h_a{}^b D_c V_b \\ =\;& f'|A|^2 X_a + (f - f'H)h_a{}^b X_b + h_a{}^b U_b - 2f' D_a h_{bc}(Y^{bc} + cg^{bc}) \\ &+ \frac{\partial}{\partial t}(\ln c)(X_a - D_a f) + \frac{f''}{(f')^2} D_a f (W - cf'H)\end{aligned}$$

and finally

(4.20)
$$
\begin{aligned}
(D_t - f'\Delta)X_a =& f'|A|^2X_a + (f - f'H){h_a}^bX_b + {h_a}^bU_b - 2f'D_ah_{bc}Y^{bc}\\
&+ \frac{\partial}{\partial t}(\ln c)X_a + \frac{f''}{(f')^2}D_afW - (2c + \frac{\partial}{\partial t}(\ln c) + c\frac{f''}{f'}H)D_af
\end{aligned}
$$

Next we compute

$$
\begin{aligned}
(D_t - f'\Delta)D_tf =& [D_t, f'\Delta]f + D_t(ff'|A|^2)\\
=& f'[D_t,\Delta]f + D_tf'\Delta f + D_tf'f|A|^2 + f'|A|^2D_tf + ff'D_t|A|^2\\
=& (f - f'H)|Df|^2 + 2ff'h_{ab}D^aD^bf + 2f'h_{ab}D^afD^bf\\
&+ \frac{f''}{(f')^2}(D_tf)^2 + f'|A|^2D_tf + 2ff'h^{ab}D_th_{ab}
\end{aligned}
$$

giving

(4.21)
$$
\begin{aligned}
(D_t - f'\Delta)D_tf =& f'|A|^2D_tf + 4ff'h^{ab}D_th_{ab} - 2f^2f'C + \frac{f''}{(f')^2}(D_tf)^2\\
&+ (f - f'H)|Df|^2 + 2f'h_{ab}D^afD^bf
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
(D_t - f'\Delta)(cf'H) =& \frac{\partial}{\partial t}(\ln c)cf'H + c\frac{f''}{f'}HD_tf + cD_tf\\
&- cf'(H(f''\Delta H + f'''|DH|^2) + f'\Delta H + 2f''|DH|^2)\\
=& \frac{\partial}{\partial t}(\ln c)cf'H + c(f''H + f')(\Delta f + f|A|^2)\\
&- c(f''H + f')(f'\Delta H) - cf'(Hf''' + 2f'')|DH|^2\\
=& \frac{\partial}{\partial t}(\ln c)cf'H + cf(f' + f''H)|A|^2 + c((f'')^2H - f'f'' - f'f'''H)|DH|^2
\end{aligned}
$$

which gives

$$
(4.22)\quad (D_t - f'\Delta)(cf'H) = \frac{\partial}{\partial t}(\ln c)cf'H + cf(f' + f''H)|A|^2 - c\big(\frac{f''}{f'}H\big)'|Df|^2
$$

(4.21), (4.22) and (4.18) give

$$
\begin{aligned}
(D_t - f'\Delta)W =& f'|A|^2 D_t f + 4ff'h^{ab}D_t h_{ab} - 2f^2 f'C + \frac{f''}{(f')^2}(D_t f)^2 + (f - f'H)|Df|^2 \\
&+ 2f'h_{ab}D^a f D^b f + \frac{\partial}{\partial t}(\ln c)cf'H + cf(f' + f''H)|A|^2 - c\big(\frac{f''}{f'}H\big)'|Df|^2 \\
&+ V^a(f'|A|^2 D_a f + 2ff'h^{bc}D_a h_{bc} + f'h_a{}^n h_n{}^l D_l f \\
&+ (f - f'H)h_a{}^b D_b f + \frac{f''}{(f')^2}D_a f D_t f) \\
&+ D^a f(U_a + \frac{\partial}{\partial t}(\ln c)V_a - f'h_{ab}D^b f) - 2f'D^a V^b D_a D_b f \\
=& f'|A|^2 W - c((f')^2 H - ff' - ff''H)|A|^2 + \frac{\partial}{\partial t}(\ln c)W - \frac{\partial}{\partial t}(\ln c)D_t f \\
&+ \frac{f''}{(f')^2}D_t f W - c\frac{f''}{f'}HD_t f + (f - f'H)\langle X, Df\rangle \\
&+ f'(4fh^{ab} - 2D^a V^b)D_t h_{ab} + 2ff'D^a V^b h_a{}^n h_{nb} - 2f^2 f'C \\
&- c\big(\frac{f''}{f'}H\big)'|Df|^2 + 2ff'V^a D_a h_{bc}h^{bc} + \langle Df, U\rangle + f'h_{ab}X^a D^b f \\
=& f'|A|^2 W - c((f')^2 H - ff' - ff''H)|A|^2 + \frac{\partial}{\partial t}(\ln c)W \\
&- (c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c))D_t f + \frac{f''}{(f')^2}D_t f W + (f - f'H)\langle X, Df\rangle \\
&+ f'(4fh^{ab} - 2D^a V^b)W_{ab} - f'(4fh^{ab} - 2D^a V^b)(D^l h_{ab}V_l + ch_{ab}) \\
&+ 2ff'D^a V^b h_a{}^n h_{nb} - 2f^2 f'C - c\big(\frac{f''}{f'}H\big)'|Df|^2 \\
&+ 2ff'V^a D_a h_{bc}h^{bc} + f'\langle h_{ab}, D_a f X_b\rangle + \langle Df, U\rangle
\end{aligned}
$$

and after rearranging terms we conclude

(4.23)

$$
\begin{aligned}
(D_t - f'\Delta)W =& f'|A|^2 W + f'(4fh^{ab} - 2D^a V^b)W_{ab} + f'h^{ab}X_a D_b f + \langle Df, U\rangle \\
&+ 2f'(fh_a{}^n h_{nb} + D^l h_{ab}V_l)Y^{ab} + (2c + \frac{\partial}{\partial t}(\ln c))W \\
&- c((f')^2 H - ff' - ff''H)|A|^2 - (c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)D_t f \\
&+ \frac{f''}{(f')^2}D_t f W + (f - f'H)\langle X, Df\rangle - c\big(\frac{f''}{f'}H\big)'|Df|^2 + 2cf'h_{ab}Y^{ab}
\end{aligned}
$$

Eventually the evolution equation for Z is given by

$$\begin{aligned}(D_t - f'\Delta)Z =& f'|A|^2W + f'(4fh^{ab} - 2D^aV^b)W_{ab} + f'h^{ab}X_aD_bf + \langle Df, U\rangle\\
&+ 2f'(fh_a{}^nh_{nb} + D^lh_{ab}V_l)Y^{ab} + (2c + \frac{\partial}{\partial t}(\ln c))W\\
&- c((f')^2H - ff' - ff''H)|A|^2 - (c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)D_tf\\
&+ \frac{f''}{(f')^2}D_tfW + (f - f'H)\langle X, Df\rangle - c\big(\frac{f''}{f'}H\big)'|Df|^2 + 2cf'h_{ab}Y^{ab}\\
&+ V^a(f'|A|^2X_a + (f - f'H)h_a{}^bX_b + h_a{}^bU_b - 2f'D_ah_{bc}Y^{bc}\\
&+ \frac{\partial}{\partial t}(\ln c)X_a + \frac{f''}{(f')^2}D_afW - (2c + \frac{\partial}{\partial t}(\ln c) + c\frac{f''}{f'}H)D_af)\\
&+ X^a(U_a + \frac{\partial}{\partial t}(\ln c)V_a - f'h_{ab}D^bf) - 2f'D_aV_b(W^{ab} + h^{an}Y_n{}^b)\\
=& f'|A|^2Z + 2\langle X, U\rangle - 4f'(Y^{ab} + cg^{ab})W_{ab} - 2f'h^{an}Y_n{}^b(Y_{ab} + cg_{ab})\\
&+ (f - f'H)|X|^2 + \frac{f''}{(f')^2}W(D_tf + \langle V, Df\rangle) - (c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)(D_tf + \langle Df, V\rangle)\\
&+ 2\frac{\partial}{\partial t}(\ln c)\langle X, V\rangle + (2c + \frac{\partial}{\partial t}(\ln c))W - c((f')^2H - ff' - ff''H)|A|^2\\
&- c\big(\frac{f''}{f'}H\big)'|Df|^2 + 2cf'h_{ab}Y^{ab}\\
=& f'|A|^2Z + 2\langle X, U\rangle - 2f'h_a{}^nY_{nb}Y^{ab} - 4f'W_{ab}Y^{ab} - 4cW\\
&+ (f - f'H)|X|^2 + \frac{f''}{(f')^2}W(W - cf'H) - (c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)(W - cf'H)\\
&+ 2\frac{\partial}{\partial t}(\ln c)\langle X, V\rangle + (2c + \frac{\partial}{\partial t}(\ln c))W - c((f')^2H - ff' - ff''H)|A|^2 - c\big(\frac{f''}{f'}H\big)'|Df|^2\end{aligned}$$

So we finally arrive at

(4.24)

$$\begin{aligned}(D_t - f'\Delta)Z =& \big(f'|A|^2 + \frac{f''}{(f')^2}(Z - 2\langle X, V\rangle) - 2(c\frac{f''}{f'}H + 2c)\big)Z\\
&+ 2\langle X, U\rangle + (f - f'H)|X|^2 + \frac{f''}{(f')^2}\langle X, V\rangle^2 + 2(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)\langle X, V\rangle\\
&- 2f'h_a{}^nY_{nb}Y^{ab} - 4f'W_{ab}Y^{ab} + (c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)cf'H\\
&+ c(ff''H + ff' - (f')^2H)|A|^2 - c\big(\frac{f''}{f'}H\big)'|Df|^2\end{aligned}$$

Now we are able to prove Theorem 1.

Proof of Theorem 1. From (4.24) and the assumptions in Theorem 1 we conclude, with $c := \frac{1}{d+(a+2)t}$

$$\begin{aligned}(D_t - f'\Delta)Z \geq& \big(f'|A|^2 + \frac{f''}{(f')^2}(Z - 2\langle X, V\rangle) - 2(c\frac{f''}{f'}H + 2c)\big)Z - 2f'h_a{}^nY_{nb}Y^{ab} - 4f'W_{ab}Y^{ab}\\
&+ 2\langle X, U\rangle + (f - f'H)|X|^2 + \frac{f''}{(f')^2}\langle X, V\rangle^2 + 2(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)\langle X, V\rangle\end{aligned}$$

Now choose d so small such that $Z > \epsilon > 0$ for $t = 0$ and for all tangent vectors V. This is possible since the initial surface is convex. On any compact time interval $[0, t_0]$ with $t_0 < T$ we can therefore estimate

$$\begin{aligned}(D_t - f'\Delta)Z \geq & -bZ - 2f'h_a{}^n Y_{nb}Y^{ab} - 4f'W_{ab}Y^{ab} \\ & + 2\langle X, U\rangle + (f - f'H)|X|^2 + \frac{f''}{(f')^2}\langle X, V\rangle^2 + 2(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)\langle X, V\rangle\end{aligned}$$

with a large constant b depending on t_0. Additionally for $t > 0$ and an arbitrary positive constant δ

$$\begin{aligned}(D_t - f'\Delta)(e^{bt}Z + \delta t) > & e^{bt}\big(-2f'h_a{}^n Y_{nb}Y^{ab} - 4f'W_{ab}Y^{ab} + 2\langle X, U\rangle + (f - f'H)|X|^2 \\ & + \frac{f''}{(f')^2}\langle X, V\rangle^2 + 2(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)\langle X, V\rangle\big)\end{aligned}$$

If $t_1 \leq t_0$ would be the first time where $e^{bt}Z + \delta t$ would become zero at some point $x \in M^n$ and for some tangent vector V then we must have $X_a = 0$ since it is the first variation with respect to V and we can also extend V in spacetime such that $Y_{ab} = 0$. This implies a contradiction and therefore $e^{bt}Z + \delta t > 0$, $\forall t \leq t_0$. Since δ and t_0 are arbitrary we conclude $Z \geq 0$ whenever $t < T$ and $d + (a+2)t > 0$. □

Now we want to answer the question for which f one can expect selfsimilar solutions of $(\star)$. First we remark that if $x \in \Omega$ and $x > 0$, then the sphere of constant radius $\frac{n}{x}$ and with constant mean curvature $H = x$ gives always rise to a selfsimilar solution for a short time.

For any subset $A \subset M^n$ let us define

$$H(A) := \{x \in \Omega | \exists p \in A : H(p) = x\}$$

and

$$P_t := \{p \in M_t^n | \nabla H \neq 0\}$$

The answer to question III is then given by

Proposition IIIa. *If $P_0 \neq \emptyset$ and $F_t : M^n \to \mathbb{R}^{n+1}$ is a selfsimilar solution of $(\star)$ for a compact connected M^n, then we have $f = A\alpha x^\alpha$, $\forall x \in H(M_t)$ with nonvanishing constants A and α.*

Proof. Since F_t is selfsimilar we have $P_t = P_0$, $\forall t \in [0, T)$. Since X_a, W vanish on selfsimilar solutions so must their time derivatives. From (4.20) we conclude that on $H(P_t)$ and by continuity also on $\overline{H(P_t)}$ we must have $c\frac{f''}{f'}x + 2c + \frac{\partial}{\partial t}(\ln c) = 0$. Since $P_0 \neq \emptyset$ and M^n is connected we have $H(M^n) \subset \overline{H(P_t)}$ and we derive

$$c\frac{f''}{f'}x + 2c + \frac{\partial}{\partial t}(\ln c) = 0$$

for all $x \in H(M_t^n)$. Since $f' > 0$ there can be at most one point $x \in \Omega$ such that $f(x) = 0$. At all other points we have

$$ff''x + ff' - (f')^2 x = f^2\big(\frac{f'}{f}x\big)'$$

Then (4.23) implies that also

$$(\frac{f'}{f}x)' = 0$$

in all points $x \in H(M_t)$ where $f(x) \neq 0$. But after integration we obtain that at these points

$$f = A\alpha x^\alpha$$

with constants A, α (nonvanishing since we must have $f' > 0$) and again by continuity this also holds on all of $H(M_t)$. □

We observe that by (2.9)

$$\Delta |F|^2 = 2(n - H\langle F, \nu\rangle)$$

and with (H.1) we obtain on a homothetic solution

$$\Delta |F|^2 = 2(n - \frac{fH}{c}) \tag{H.8}$$

This implies

Proposition IIIb. *If $P_0 = \emptyset$ and M^n is a compact orientable selfsimilar solution of $(\star)$, then M_t is a sphere of radius $\frac{n}{H}$. If $f = -\frac{1}{H}$, then any compact orientable homothetic solution is a sphere of radius $\frac{n}{H}$.*

Proof. $P_0 = \emptyset$ and (H.8) imply $\Delta |F|^2 = const$. By the assumptions on M^n this constant must be zero and then consequently $|F|^2 = const$. This implies that M_t is a sphere. □

5. Longtime existence for some highly nonlinear flows

In this paragraph we are going to show that one can use the Harnack inequality to prove longtime existence of solutions for some highly nonlinear flows. To be precise

Theorem 2. *Assume that $f : (0, \infty) \to \mathbb{R}$ is a smooth negative function that satisfies the assumptions in Theorem 1 with $a > -2$ and that $\lim_{x\to 0} f(x) = -\infty$ and $F_0 : M^2 \to \mathbb{R}^3$ is a smooth convex immersion of an orientable compact surface M^2. Then $(\star)$ has a smooth immortal solution and we have*

$$\lim_{t\to\infty} H = 0 \ , \ \lim_{t\to\infty} |F_t|^2 = \infty$$

Remark. The functions $f = \alpha x^\alpha$ with $-1 < \alpha < 0$ satisfy the assumptions in Theorem 2 on $\Omega = (0, \infty)$. These speed functions are not homogenous of degree one and are not included in the class of functions considered in [2]

To prove Theorem 2 we need some lemmas that are interesting on their own.

Lemma 1. *Assume $f' > 0$ and that $F_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth immersion of an orientable compact surface and that on $M_0 = F_0(M^n)$ we have $f^2 \geq \min_{M_0} f^2 > 0$. Then this is also true on the maximal time interval $[0, T)$ where a smooth solution of $(\star)$ exists.*

Proof. The evolution equation for f^2 is given by

$$(D_t - f'\Delta)f^2 = f'\big(-2|\nabla f|^2 + 2f^2|A|^2\big)$$

and the result follows from the parabolic maximum principle. $\square$

Corollary 1. *With the assumptions in Lemma* 1 *we have that for negative* f

$$H \le \max_{M_0} H$$

and for positive f

$$H \ge \min_{M_0} H$$

We can do even better

Lemma 2. *Assume* $\Omega = (0,\infty)$, $f < 0$, $f' > 0$ *and that* $F_0 : M^n \to \mathbb{R}^{n+1}$ *is an admissible immersion of an orientable compact manifold* M^n. *Then we can find a positive* ϵ *such that on* $[0,T)$

$$\max_{M_t} H \le \frac{\max_{M_0} H}{1 + \epsilon t \max_{M_0} H}$$

Proof. Since F_t is admissible on $[0,T)$ we must always have $H > 0$. At a point where $\max_{M_t} H$ is attained we have $\Delta H \le 0$ and $\nabla H = 0$. The evolution equation for H (2.6), Lemma 1 and the fact that $|A|^2 \ge \frac{H^2}{n}$ give us

$$\frac{\partial}{\partial t}\max_{M_t} H \le -\epsilon(\max_{M_t} H)^2$$

with $\epsilon = \frac{-\max_{M_0} f}{n}$ and after integration we obtain the result. $\square$

(2.7) and Simons identity give

$$(D_t - f'\Delta)|A|^2 = -2f'|\nabla A|^2 + 2f'|A|^4 + 2(f - f'H)C + 2f''h_{ij}\nabla^i H\nabla^j H \tag{5.1}$$

Let $R = H^2 - |A|^2$ be the Scalar curvature of M_t. Now we turn our attention to the case where $n = 2$. In this case we can decompose C into

$$C = \frac{H}{2}(3|A|^2 - H^2)$$

Then (5.1) and (2.6) imply that for $n = 2$

(5.2)
$$\begin{aligned}(D_t - f'\Delta)R =& 2f'(|\nabla A|^2 - |\nabla H|^2) + 2f''(H|\nabla H|^2 - h_{ij}\nabla^i H\nabla^j H)\\ &- (2f'|A|^2 + (f - f'H)H)R\end{aligned}$$

Lemma 3. *Under the assumptions in Theorem 2 we have*

$$R > 0,\ \forall t \in [0,T)$$

Proof. Let $t_0 \leq T$ be the maximal time such that $R > 0$ on all of M_t for $t \in [0,t_0)$, i.e., the maximal time for which M_t stays convex. If $t_0 = T$ we are done. So assume that $t_0 < T$. Since M_t is admissible for $t \in [0,t_0]$ we must have $H > 0$ on $[0,t_0]$. By definition of t_0 we also have $R > 0$ on $[0,t_0)$ and $R \geq 0$ on $[0,t_0]$. Since M and $[0,t_0]$ are both compact and $R \geq 0$ we can find a constant c such that

$$(5.3)\quad (D_t - f'\Delta)R \geq 2f'(|\nabla A|^2 - |\nabla H|^2) + 2f''(H|\nabla H|^2 - h_{ij}\nabla^i H \nabla^j H) - cR$$

Now assume that x is any point on M_t where the minimum of R is attained. At this point we must have $f'\Delta R \geq 0$ and $\nabla R = 0$. Let α and β denote the two principal curvatures in a neighborhood of x. We have

$$0 = \nabla R = 2\alpha\nabla\beta + 2\beta\nabla\alpha$$

Since $H > 0$ and $\alpha, \beta \geq 0$ we must either have $\alpha > 0$ or $\beta > 0$. Let us assume that $\alpha > 0$. First we compute that

$$H|\nabla H|^2 - h_{ij}\nabla^i H\nabla^j H = \alpha|\nabla_2 H|^2 + \beta|\nabla_1 H|^2 \geq 0,$$

where we choose normal coordinates such that $h_{ij} = \operatorname{diag}(\alpha,\beta)$. Since $f''x \geq af'$ and $H > 0, a > -2$ we can estimate

$$2f''(H|\nabla H|^2 - h_{ij}\nabla^i H\nabla^j H) \geq -4\frac{f'}{H}(\alpha|\nabla_2 H|^2 + \beta|\nabla_1 H|^2)$$

On the other hand Codazzi's equation gives us that

$$|\nabla A|^2 = |\nabla_1\alpha|^2 + |\nabla_2\beta|^2 + 3|\nabla_2\alpha|^2 + 3|\nabla_1\beta|^2$$

Combining the last two statements we see that at a point where $\nabla R = 0$ we must always have

$$2f'(|\nabla A|^2 - |\nabla H|^2) + 2f''(H|\nabla H|^2 - h_{ij}\nabla^i H\nabla^j H) \geq 0$$

and consequently at any point where the minimum of R is attained

$$D_t R \geq -cR$$

which implies that

$$\min_{M_t} R \geq (\min_{M_0} R)e^{-ct} > 0,\ \forall t \in [0,t_0]$$

This proves that $t_0 = T$. $\square$

We come to the proof of Theorem 2

Proof of Theorem 2. On $[0,T)$ we must have $H>0$: Otherwise $(\star)$ is not well-defined, i.e., the surfaces would not be admissible. If T is finite, then this can only happen for two reasons. Either the solutions converge to a surface that is no longer admissible, i.e., the mean curvature would vanish somewhere, or the surfaces must develop a singularity. Since M_t is convex on $[0,T)$ we can apply Theorem 1 with $V=0$ to obtain

$$\frac{\partial}{\partial t}H \geq -\frac{1}{d+(a+2)t}H$$

and since by assumption $a>-2$ we can estimate

$$\frac{\partial}{\partial t}H \geq -\frac{1}{d}H$$

which implies that

$$H \geq e^{\frac{-1}{d}t}\min_{M_0} H$$

So H cannot become zero in finite time. This estimate and Corollary 1 imply that the surfaces stay admissible in finite time. Therefore we conclude that if T is finite the surfaces must develop a singularity. It is well-known that the second fundamental form $|A|^2$ must then blow up for $t\to T$ (compare [2] and [5]). By Lemma 3 we conclude that in the case where $T<\infty$, H must also blow up which, in view of Corollary 1, proves that $T=\infty$. By Lemma 2 we conclude that $\lim_{t\to\infty}(\max_{M_t} H)=0$. It remains to prove that $|F_t|\to\infty$. An easy calculation gives the following evolution equation

$$(D_t - f'\Delta)(\frac{\langle F,\nu\rangle}{f}+(2+a)t) = 2\frac{f'}{f}\langle\nabla(\frac{\langle F,\nu\rangle}{f}+(2+a)t),\nabla f\rangle + a+1-\frac{f'H}{f}$$

The assumptions on f imply that $\lim_{x\to 0}\frac{f''x}{f'}$ exists and that $\lim_{x\to 0}\frac{f''x}{f'}\geq a$. Since $\lim_{x\to 0} f = -\infty$ we can apply de l'Hospital's rule and get $\lim_{x\to 0}\frac{f'x}{f} = 1+\lim_{x\to 0}\frac{f''x}{f'}\geq a+1$. On the other hand the assumption that $\left(\frac{f'x}{f}\right)'\geq 0$ implies for any $x\in\Omega$

$$a+1-\frac{f'x}{f}\leq 0$$

This gives

$$(D_t - f'\Delta)(\frac{\langle F,\nu\rangle}{f}+(2+a)t) \leq 2\frac{f'}{f}\langle\nabla(\frac{\langle F,\nu\rangle}{f}+(2+a)t),\nabla f\rangle$$

and with the maximum principle we conclude

$$\frac{\langle F,\nu\rangle}{f}\leq c-(2+a)t$$

for a constant c. Using Schwarz' inequality we have proven that

$$\frac{|F|}{|f|}\geq (2+a)t-c$$

and since $\lim_{t\to\infty}|f|\to\infty$ and $2+a>0$, this can only hold when $\lim_{t\to\infty}|F|=\infty$, proving the rest of Theorem 2. $\square$

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