New York Journal of Mathematics

New York J. Math. **3** (1997) 48–53.

A Refinement of Ball's Theorem on Young Measures

Norbert Hungerbühler

Abstract. For a sequence $u_j : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ generating the Young measure $\nu_x, x \in \Omega$, Ball's Theorem asserts that a tightness condition preventing mass in the target from escaping to infinity implies that ν_x is a probability measure and that $f(u_k) \rightharpoonup \langle \nu_x, f\rangle$ in L^1 provided the sequence is equiintegrable. Here we show that Ball's tightness condition is necessary for the conclusions to hold and that in fact all three, the tightness condition, the assertion $\|\nu_x\| = 1$, and the convergence conclusion, are equivalent. We give some simple applications of this observation.

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1. Introduction

Young measures have in recent years become an increasingly indispensable tool in the calculus of variations and in the theory of nonlinear partial differential equations (see, e.g., [5] or [2]. For a list of references for general Young measure theory see, e.g., [6]). In [1], Ball stated the following version of the fundamental theorem of Young measures, tailored for applications in these fields:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $u_j \colon \Omega \to \mathbb{R}^m$, $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $u_j \to K$ in measure as $j \to \infty$, i.e., given any open neighbourhood U of K in $\mathbb{R}^m$*

$$\lim_{j\to\infty} |\{x \in \Omega \colon u_j(x) \notin U\}| = 0.$$

Then there exists a subsequence u_k of u_j and a family (ν_x), $x \in \Omega$, of positive measures on $\mathbb{R}^m$, depending measurably on x, such that

(i) $\|\nu_x\|_{\mathcal{M}} := \int_{\mathbb{R}^m} d\nu_x \le 1$ *for a.e.* $x \in \Omega$,

Received April 9, 1997.
Mathematics Subject Classification. 46E27, 28A33, 28A20.
Key words and phrases. Young measures.

ISSN 1076-9803/97

(ii) $\operatorname{spt}\nu_x \subset K$ *for a.e.* $x \in \Omega$*, and*

(iii) $f(u_k) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f(\lambda)d\nu_x(\lambda)$ *in* $L^\infty(\Omega)$ *for each continuous function* $f\colon \mathbb{R}^m \to \mathbb{R}$ *satisfying* $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

Suppose further that $\{u_k\}$ *satisfies the boundedness condition*

$$\forall R > 0\colon \lim_{L\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_R\colon |u_k(x)| \geq L\}| = 0, \tag{1}$$

where $B_R = B_R(0)$*. Then*

$$\|\nu_x\|_{\mathcal{M}} = 1 \qquad \text{for a.e. } x \in \Omega \tag{2}$$

(*i.e.,* ν_x *is a probability measure*)*, and the following condition holds:*

$$\begin{cases} \text{For any measurable } A \subset \Omega \text{ and any continuous function } f\colon \mathbb{R}^m \to \mathbb{R} \text{ such} \\ \text{that } \{f(u_k)\} \text{ is sequentially weakly relatively compact in } L^1(A) \text{ we have} \\ f(u_k) \rightharpoonup \langle \nu_x, f\rangle \text{ in } L^1(A). \end{cases} \tag{3}$$

Improved versions of this theorem can be found, e.g., in [4]. The aim of this note is to prove that (1) is necessary for (2) and (3) to hold, and that in fact (1), (2) and (3) are equivalent. We will give some simple consequences of this fact.

Theorem 2. *Let* Ω*,* u_j *and* ν_x *be as in Theorem 1. Then (1), (2) and (3) are equivalent.*

Remarks: (a) It was proved in [1] that (1) is equivalent to the following condition: Given any $R > 0$ there exists a continuous nondecreasing function $g_R\colon [0,\infty) \to \mathbb{R}$, with $\lim_{t\to\infty} g_R(t) = \infty$, such that

$$\sup_{k\in\mathbb{N}} \int_{\Omega\cap B_R} g_R(|u_k(x)|)dx < \infty.$$

(b) In [1] it is also shown, that under hypothesis (1) for any measurable $A \subset \Omega$

$$f(\cdot, u_k) \rightharpoonup \langle \nu_x, f(x,\cdot)\rangle \quad \text{in } L^1(A)$$

for every Carathéodory function $f\colon A \times \mathbb{R}^m \to \mathbb{R}$ such that $\{f(\cdot, u_k)\}$ is sequentially weakly relative compact in $L^1(A)$. Hence, this fact is also equivalent to (1), (2) and (3).

(c) Ball also shows in [1], that if u_k generates the Young measure ν_x, then for $\psi \in L^1(\Omega; C_0(\mathbb{R}^m))$

$$\lim_{k\to\infty} \int_\Omega \psi(x, u_k(x))dx = \int_\Omega \langle \nu_x, \psi(x,\cdot)\rangle dx.$$

Here, $C_0(\mathbb{R}^m)$ denotes the Banach space of continuous functions $f\colon \mathbb{R}^m \to \mathbb{R}$ satisfying $\lim_{|\lambda|\to\infty} f(\lambda) \to 0$ equipped with the L^∞-norm.

2. Proof of Theorem 2

First we prove (2) $\Longrightarrow$ (1). We assume by contradiction that (2) holds and that there exists $R > 0$ and $\varepsilon > 0$ with the following property: There exists a sequence $L_i \to \infty$ and integers k_i such that $|\{x \in \Omega \cap B_R\colon |u_{k_i}(x)| \geq L_i\}| > \varepsilon$ for all $i \in \mathbb{N}$. For $\rho > 0$ consider the function

$$\alpha_\rho(t) := \begin{cases} 1 & \text{if } t \leq \rho \\ 0 & \text{if } t \geq \rho + 1 \\ \rho + 1 - t & \text{if } \rho < t < \rho + 1 \end{cases}$$

Then $\varphi_\rho\colon \mathbb{R}^m \to \mathbb{R}, x \mapsto \alpha_\rho(|x|)$, is in $C_0(\mathbb{R}^m)$. Hence, applying the first part of Theorem 1, we have that

$$\lim_{k\to\infty}\int_\Omega \varphi_\rho(u_k)\chi_{B_R}dx = \int_\Omega\int_{\mathbb{R}^m}\varphi_\rho(\lambda)d\nu_x(\lambda)\chi_{B_R}dx. \tag{4}$$

Notice that $k_i \to \infty$ for $i \to \infty$ since the functions u_j are finite for a.e. $x \in \Omega$. Hence, u_{k_i} is a subsequence of u_k and for i large enough, we find

$$|\Omega \cap B_R| - \varepsilon \geq \int_\Omega \varphi_\rho(u_{k_i})\chi_{B_R}dx.$$

Thus, (4) implies

$$|\Omega \cap B_R| - \varepsilon \geq \int_\Omega\int_{\mathbb{R}^m}\varphi_\rho(\lambda)d\nu_x(\lambda)\chi_{B_R}dx. \tag{5}$$

On the other hand, by the monotone convergence theorem, we conclude that the right hand side of (5) converges for $\rho \to \infty$ to

$$\int_\Omega\int_{\mathbb{R}^m} d\nu_x(\lambda)\chi_{B_R}dx = \int_\Omega \|\nu_x\|_{\mathcal{M}}\chi_{B_R}dx = |\Omega \cap B_R|$$

by (2) and this contradicts (5).

Second we prove that (3) $\implies$ (2). Let $R > 0$ be fixed and let f denote the function constant 1 on $\mathbb{R}^m$. Then $f(u_j)$ is sequentially weakly relative compact on $\Omega \cap B_R$ and (3) implies

$$|\Omega \cap B_R| = \int_{\Omega\cap B_R} f(u_k)\chi_{B_R}dx \to \tag{6}$$
$$\to \int_{\Omega\cap B_R}\int_{\mathbb{R}^m} f(\lambda)d\nu_x(\lambda)\chi_{B_R}dx = \int_{\Omega\cap B_R}\|\nu_x\|_{\mathcal{M}}dx.$$

Since $\|\nu_x\|_{\mathcal{M}} \leq 1$ by (i) in Theorem 1, we conclude that $\|\nu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega \cap B_R$. Since R was arbitrary, the claim follows.

3. Applications

The following propositions are certainly known to the experts in the field, but we want to show that the sharp version of Ball's theorem which we now have at our disposal, considerably simplifies the proofs.

Proposition 1. *If* $|\Omega| < \infty$ *and* ν_x *is the Young measure generated by the* (*whole*) *sequence* u_j *then*

$$u_j \to u \text{ in measure} \iff \nu_x = \delta_{u(x)} \text{ for a. e. } x \in \Omega.$$

A precursor of this proposition can be found, e.g., in [3] (see also [4]).

Proof. Let us first assume that $u_j \to u$ in measure, i.e., $\forall\varepsilon > 0$ we have

$$\lim_{j\to\infty}|\{|u_j - u| > \varepsilon\}| = 0. \tag{7}$$

For $\varphi \in C_c^\infty(\mathbb{R}^m)$ and $\zeta \in L^1(\Omega)$,

$$\left|\int_\Omega \zeta\,(\varphi(u_j) - \varphi(u))dx\right| \leq$$
$$\left|\int_{|u_j-u|>\varepsilon} \zeta\,(\varphi(u_j) - \varphi(u))dx\right| + \left|\int_{|u_j-u|\leq\varepsilon} \zeta\,(\varphi(u_j) - \varphi(u))dx\right| =: I + II.$$

By choosing ε appropriately, we can make II as small as we want, since we observe that $II \leq \varepsilon\|D\varphi\|_{L^\infty}\|\zeta\|_{L^1}$. For I we then have

$$I \leq 2\|\varphi\|_{L^\infty} \int_{|u_j-u|>\varepsilon} |\zeta| dx$$

which converges to 0 as j tends to ∞ by absolute continuity of the integral and (7). Since C_c^∞ is dense in C_0 we conclude that for all $\varphi \in C_0$

$$\varphi(u_j) \overset{*}{\rightharpoonup} \langle \delta_{u(x)}, \varphi\rangle \quad \text{in } L^\infty(\Omega)$$

and hence $\nu_x = \delta_{u(x)}$.

Now we assume $\nu_x = \delta_{u(x)}$, hence (2) is fulfilled. First step: We consider the case that u_j is bounded in L^∞. Then by (3) we conclude that for $\varphi(x) := |x|^2$

$$\|u_j\|_{L^2}^2 = \int_\Omega \varphi(u_j)dx \to \int_\Omega \varphi(u)dx = \|u\|_{L^2}^2 \tag{8}$$

for $j \to \infty$. On the other hand choosing $\varphi = \text{id}$ we similarly find that $u_j \rightharpoonup u$ weakly in $L^2(\Omega)$, which in combination with (8) gives that $u_j \to u$ in $L^1(\Omega)$. Thus for all $\alpha > 0$ we have

$$\alpha|\{|u_j - u| \geq \alpha\}| \leq \int_\Omega |u_j - u| dx \to 0$$

as $j \to \infty$, and hence $u_j \to u$ in measure.

Second step: We show that if u_j generates the Young measure $\delta_{u(x)}$ then $T_R(u_j) \to T_R(u)$ in measure, where T_R denotes the truncation $T_R(x) := x \min\{1, \frac{R}{|x|}\}$, $R > 0$ fixed. In fact, for $f \in C_0(\mathbb{R}^m)$ we have that $f \circ T_R$ is continuous and $f(T_R(u_j))$ is equiintegrable (and hence, by the Dunford-Pettis theorem, sequentially weakly precompact in $L^1(\Omega)$). Since (2) is fulfilled, we conclude by (3) that for $\zeta \in L^\infty(\Omega)$

$$\int_\Omega \zeta\, f(T_R(u_j))dx \to \int_\Omega \zeta\, f(T_R(u))dx.$$

This implies that $T_R(u_j)$ generates the Young measure $\delta_{T_R(u(x))}$ and by the first step, the claim follows.

Third step: We show, that $u_j \to u$ in measure. Let $\varepsilon > 0$ be given. Then we have:

$$\begin{aligned}|\{|u_j - u| > \varepsilon\}| &\leq |\{|u_j - u| > \varepsilon, |u| \leq R, |u_j| \leq R\}| + |\{|u| > R\}| + |\{|u_j| > R\}| \\ &=: I + II + III.\end{aligned}$$

II can be made arbitrarily small by choosing $R > 0$ large enough. By (2) we have (1) which implies that III is, again for R large enough, uniformly in j as small as we want. Finally by the second step, $I \to 0$ for $j \to \infty$. □

Our second application is the following proposition:

Proposition 2. *Let $|\Omega| < \infty$. If the sequences $u_j\colon \Omega \to \mathbb{R}^m$ and $v_j\colon \Omega \to \mathbb{R}^k$ generate the Young measures $\delta_{u(x)}$ and ν_x respectively, then (u_j, v_j) generates the Young measure $\delta_{u(x)} \otimes \nu_x$.*

This result also holds for sequences μ_j, λ_j of Young measures converging in the narrow topology to μ and λ respectively: see [6]. However it is false if both μ and λ are not Dirac measures. E.g., consider the Rademacher functions $u_1(x) := (-1)^{\lfloor x \rfloor}$ and $u_n(x) = u_1(nx)$. u_n and $-u_n$ generate the Young measure $\frac{1}{2}(\delta_{-1} + \delta_1)$, but (u_n, u_n) and $(-u_n, u_n)$ obviously generate different measures (consider the sets $K = \{(-1,-1),(1,1)\}$ and $K = \{(-1,1),(1,-1)\}$ respectively in Theorem 1).

Proof of Proposition 2. We have to show that for all $\varphi \in C_c^\infty(\mathbb{R}^m \times \mathbb{R}^k)$ there holds $\varphi(u_j, v_j) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^k} \varphi(u(x), \lambda) d\nu_x(\lambda)$. So, let $\zeta \in L^1(\Omega)$. We have

$$\begin{aligned}
&\left| \int_\Omega \zeta \left(\varphi(u_j, v_j) - \int_{\mathbb{R}^k} \varphi(u, \lambda) d\nu_x(\lambda) \right) dx \right| \leq \\
&\leq \left| \int_{|u_j - u| < \varepsilon} \zeta \left(\varphi(u_j, v_j) - \varphi(u, v_j) \right) dx \right| + \left| \int_{|u_j - u| \geq \varepsilon} \zeta \left(\varphi(u_j, v_j) - \varphi(u, v_j) \right) dx \right| \\
&\qquad + \left| \int_\Omega \zeta \left(\varphi(u, v_j) - \int_{\mathbb{R}^k} \varphi(u, \lambda) d\nu_x(\lambda) \right) dx \right| =: I + II + III.
\end{aligned}$$

Since $I \leq \varepsilon \|\zeta\|_{L^1(\Omega)} \|D\varphi\|_{L^\infty}$, the first term is small for $\varepsilon > 0$ small. For $\varepsilon > 0$ fixed, we have for $j \to \infty$

$$II \leq 2\|\varphi\|_{L^\infty} \int_{|u_j - u| \geq \varepsilon} |\zeta| dx \to 0$$

since by Proposition 1 the sequence u_j converges to u in measure. Since $L^\infty(\Omega)$ is dense in $L^1(\Omega)$ we may assume that $\zeta \in L^\infty(\Omega)$. Thus, the function $\zeta(x)\varphi(u(x), \cdot)$ is in $L^1(\Omega, C_0(\mathbb{R}^k))$ and hence $III \to 0$ as $j \to \infty$ by Remark (c). □

The third application we consider is a criterion for the pointwise convergence of Fourier series, which is similar to Dini's test (see [7]).

Theorem 3. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a 2π periodic complex function. If $z \in \mathbb{R}$ is a point with the property that*

$$\int_{-\pi}^{\pi} \left| \frac{f(x) - f(z)}{x - z} \right| dx < \infty \tag{9}$$

then the Fourier series of f converges in z to $f(z)$.

Proof. With

$$D_N(x) := \frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$$

the Nth Fourier approximation of f is $s_N(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z-x) D_N(x) dx$. The Young measure generated by the (whole) sequence $\sin(N + \frac{1}{2})x$ is a probability measure ν_x with vanishing first moment $\langle \nu_x, \mathrm{id} \rangle = 0$ (see, e.g., [6]). Hence, the Young measure μ_x generated by the sequence $(f(z-x) - f(z)) D_N(x)$ also has these properties for a. e. x. Now, (9) implies that the sequence $(f(z-x) - f(z)) D_N(x)$ is equiintegrable and hence (since μ_x is a probability measure for a. e. x) by equivalence of (2) and (3), we have as $N \to \infty$

$$s_N(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(z-x)-f(z))D_N(x)dx + \frac{f(z)}{2\pi}\int_{-\pi}^{\pi}D_N(x)dx \to f(z)$$

since the first term converges to zero and the second term equals $f(z)$ for all N. □

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Departement Mathematik, ETH-Zentrum, CH-8092 Zürich (Switzerland)
buhler@math.ethz.ch, http://www.math.ethz.ch/~buhler/noebi.html