New York Journal of Mathematics

New York J. Math. **2** (1996) 59–68.

Proof of the Refined Alternating Sign Matrix Conjecture

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Abstract. Mills, Robbins, and Rumsey conjectured, and Zeilberger proved, that the number of alternating sign matrices of order n equals

$$A(n) := \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}.$$

Mills, Robbins, and Rumsey also made the stronger conjecture that the number of such matrices whose (unique) '1' of the first row is at the r^{th} column equals

$$A(n)\frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}}.$$

Standing on the shoulders of A. G. Izergin, V. E. Korepin, and G. Kuperberg, and using in addition orthogonal polynomials and q-calculus, this stronger conjecture is proved.

Contents

Introduction

An *alternating sign matrix*, or ASM, is a matrix of 0's, 1's, and −1's such that the non-zero elements in each row and each column alternate between 1 and −1

Received November 8, 1995.
Mathematics Subject Classification. Primary: 05A; Secondary: 33.
Key words and phrases. enumeration, alternating sign matrices, square ice, Izergin-Korepin formula, orthogonal polynomials, q-analysis, q-Legendre polynomials, exactly solvable models.
Supported in part by the NSF.

ISSN 1076-9803/96

and begin and end with 1, for example:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Mills, Robbins, and Rumsey [MRR1] [MRR2] ([S], Conj. 1) conjectured, and I proved [Z], that there are

$$A(n) := \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}$$

alternating sign matrices of order n. Another, shorter, proof was later given by Greg Kuperberg [K]. Kuperberg deduced the straight enumeration of ASMs from their weighted enumeration by Izergin and Korepin [KBI]. In this paper, I extend Kuperberg's method of proof to prove the more general, refined enumeration, also conjectured in [MRR1] [MRR2], and listed by Richard Stanley [S] as the third of of his "Baker's Dozen", that:

Main Theorem. *There are*

$$A(n,r) := A(n)\frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}}$$

$n \times n$ *alternating sign matrices for which the* (*unique*) '1' *of the first row is at the* r^{th} *column.*

As in Kuperberg's proof, we are reduced to evaluating a certain determinant. Unlike the original determinant, it is not evaluable in closed form. We evaluate it by using the q-analog of the Legendre polynomials over an interval, introduced, and greatly generalized, by Askey and Andrews [AA] and Askey and Wilson [AW]. All that is needed from the general theory of orthogonal polynomials and from q-calculus is reviewed, so a sufficient condition for following the present paper is having read Kuperberg's paper [K] that includes a very clear exposition of the Izergin-Korepin formula, and of its proof. Of course, this is also a necessary condition. In order to encourage readers to look up and read Kuperberg's beautiful paper [K], and to save myself some typing, I will use the notation, and results, of [K], without reviewing them.

Boiling It Down To a Determinant Identity

Let $B(n,r)$ be the number of ASMs of order n whose sole '1' of the first (equivalently last) row, is at the r^{th} column. In order to stand on Kuperberg's shoulders more comfortably, we will consider the last row rather than the first row.

As in [K], define $[x] := (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$, take $q := e^{2\sqrt{-1}\pi/3}$, and consider

$$Z(n\,;2,\ldots,2\,,2+a\,;0,\ldots,0),$$

where between the two semi-colons inside Z there are $n-1$ 2's followed by a single $2+a$. Here a is an indeterminate. Let's look at an ASM of order n, whose sole '1' of the last row is at the r^{th} column. It is readily seen that the $r-1$ zeros to the left of that '1' each contribute a weight of $[2+a]$, while the remaining $n-r$ zeros, to the right of the aforementioned '1', each contribute a weight of $[1+a]$. The '1' itself contributes $q^{-1-a/2}$, which is $q^{-a/2}$ times what it did before. Hence

$$Z(n\,;\,2,\ldots,\,2\,,\,2+a\,;\,0,\ldots,0)\;=\;(-1)^n q^{-n} q^{-a/2}\sum_{r=1}^{n} B(n,r)[2+a]^{r-1}[1+a]^{n-r}.$$

We also have, thanks to [K], (or [Z]: just plug in $a=0$ above):

$$Z(n\,;\,2,\ldots,2\,,\,2\,;\,0,\ldots,0)\;=\;(-1)^n q^{-n} A(n).$$

Hence:

$$\frac{Z(n\,;\,2,\ldots,2\,,\,2+a\,;\,0,\ldots,0)}{Z(n;2,\ldots,2,\,2\,;0,\ldots,0)}=\frac{q^{-a/2}}{A(n)}\sum_{r=1}^{n} B(n,r)[2+a]^{r-1}[1+a]^{n-r}.$$

Since

$$\{[2+a]^{r-1}[1+a]^{n-r}\,;\,1\le r\le n\}$$

are linearly independent, the $B(n,r)$ are uniquely determined by the above equation. Hence the Main Theorem is equivalent to:

$$\frac{Z(n;2,\ldots,2,2+a;0,\ldots,0)}{Z(n;2,\ldots,2,2;0,\ldots,0)}=$$

$$\frac{q^{-a/2}}{\binom{3n-2}{n-1}}\sum_{r=1}^{n}\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}[2+a]^{r-1}[1+a]^{n-r}.$$

By replacing n by $n+1$, and changing the summation on r to start at 0, we get that it suffices to prove:

$$\frac{Z(n+1;2,\ldots,2,2+a;0,\ldots,0)}{Z(n+1;2,\ldots,2,2;0,\ldots,0)}=\frac{q^{-a/2}}{\binom{3n+1}{n}}\sum_{r=0}^{n}\binom{n+r}{n}\binom{2n-r}{n}[2+a]^{r}[1+a]^{n-r}.$$

Let $\widetilde{Z}(n;x_1,\ldots,x_n;y_1,\ldots,y_n)$ denote the right hand side of the Izergin-Korepin formula (Theorem 6 of [K]). First replace n by $n+1$. Then, taking $x_i=2+i\epsilon$, for $i=1,\ldots,n$, $x_{n+1}=2+a+(n+1)\epsilon$, and $y_j=-(j-1)\epsilon$, $j=1,\ldots,n+1$, yields, after cancellation,

$$\frac{\widetilde{Z}(n+1\,;\,2+\epsilon,\ldots,2+n\epsilon\,,\,2+a+(n+1)\epsilon\,;\,0,-\epsilon,\ldots,-n\epsilon)}{\widetilde{Z}(n+1\,;\,2+\epsilon,\ldots,2+n\epsilon\,,\,2+(n+1)\epsilon\,;\,0,-\epsilon,\ldots,-n\epsilon)}=$$

$$q^{-a/2}\prod_{j=0}^{n}\frac{[2+a+(n+1+j)\epsilon][1+a+(n+1+j)\epsilon]}{[2+(n+1+j)\epsilon][1+(n+1+j)\epsilon]}\cdot\prod_{j=0}^{n-1}\frac{[(n-j)\epsilon]}{[a+(n-j)\epsilon]}\cdot\frac{\det M_{n+1}(a)}{\det M_{n+1}(0)},$$

where $M_{n+1}(a)=(m_{i,j}\,,\,0\le i,j\le n)$ is the $(n+1)\times(n+1)$ matrix, defined as follows:

$$m_{i,j}=\begin{cases}1/([2+(i+j+1)\epsilon][1+(i+j+1)\epsilon]) & \text{if } 0\le i\le n-1,\ 0\le j\le n;\\ 1/([2+a+(n+j+1)\epsilon][1+a+(n+j+1)\epsilon]) & \text{if } i=n,\ 0\le j\le n.\end{cases}$$

Taking the limit $\epsilon \to 0$, replacing q^ϵ by s, q^a by X, setting $w := e^{\sqrt{-1}\pi/3}$, evaluating the limit whenever possible, and cancelling out whenever possible, reduces our task to proving the following identity:

$$\lim_{s \to 1} \left\{ \frac{(1-s)^n \det N_{n+1}(X)}{\det N_{n+1}(1)} \right\} = \qquad \text{(Not Yet Done)}$$
$$\frac{-(1-X)^n(\sqrt{-3})^{n+2}w^{-n}}{n!(1+X+X^2)^{n+1}\binom{3n+1}{n}} \cdot \sum_{r=0}^{n} w^{-r}\binom{n+r}{n}\binom{2n-r}{n}(1+wX)^r(1-w^2X)^{n-r}.$$

Here the matrix $N_{n+1}(X)$ is $M_{n+1}(a)$ divided by 3, to wit: $N_{n+1}(X) = (p_{i,j}, 0 \leq i, j \leq n)$ is the $(n+1)\times(n+1)$ matrix, defined as follows. For the first n rows we have:

$$p_{i,j} = \frac{1-s^{i+j+1}}{1-s^{3(i+j+1)}}, \quad (\, 0 \leq i \leq n-1 \,,\ 0 \leq j \leq n \,),$$

while for the last row we have:

$$p_{n,j} = \frac{1-Xs^{n+j+1}}{1-X^3s^{3(n+j+1)}}, \quad (\, 0 \leq j \leq n \,).$$

We are left with the task of computing the determinant of $N_{n+1}(X)$, or at least the limit on the left of (Not Yet Done).

A Short Course on Orthogonal Polynomials

I will only cover what we need here. Of the many available accounts, Chapter 2 of [Wilf] is especially recommended. For the present purposes, section IV of [D] is most pertinent.

Theorem OP. *Let T be any linear functional* (*'umbra'*) *on the set of polynomials, and let $c_i := T(x^i)$ be its so-called* moments. *Let*

$$\Delta_n := \det \begin{pmatrix} c_0 & c_1 & \dots & \dots & c_n \\ c_1 & c_2 & \dots & \dots & c_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ c_n & c_{n+1} & \dots & \dots & c_{2n} \end{pmatrix}.$$

If $\Delta_n \neq 0$, for $n \geq 0$, then there is a unique sequence of monic polynomials $P_n(x)$, where the degree of $P_n(x)$ is n, that are orthogonal with respect to the functional T:

$$T(P_n(x)P_m(x)) = 0 \quad if \quad m \neq n.$$

Furthermore, these polynomials $P_n(x)$ are given 'explicitly' by:

$$P_n(x) = \frac{1}{\Delta_{n-1}} \det \begin{pmatrix} c_0 & c_1 & \dots & \dots & c_n \\ c_1 & c_2 & \dots & \dots & c_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n-1} & c_n & \dots & \dots & c_{2n-1} \\ 1 & x & x^2 & \dots & x^n \end{pmatrix}. \qquad \text{(General Formula)}$$

□

Corollary 1.

$$T(x^n P_n(x)) = \frac{\Delta_n}{\Delta_{n-1}} \quad for \quad n \geq 1.$$

□

Corollary 2. *If S is another linear functional, $d_i := S(x^i)$, and*

$$\Gamma_n := \det \begin{pmatrix} c_0 & c_1 & \dots & \dots & c_n \\ c_1 & c_2 & \dots & \dots & c_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n-1} & c_n & \dots & \dots & c_{2n-1} \\ d_0 & d_1 & d_2 & \dots & d_n \end{pmatrix},$$

then

$$\frac{\Gamma_n}{\Delta_n} = \frac{S(P_n(x))}{T(x^n P_n(x))} \quad for \quad n \geq 1.$$

□

For a long time it was believed that Theorem **OP** was only of theoretical interest, and that, given the moments, it was impractical to actually find the polynomials $P_n(x)$, by evaluating the determinant. This conventional wisdom was conveyed by Richard Askey, back in the late seventies, to Jim Wilson, who was then studying under him. Luckily, Wilson did not take this advice. Using Theorem **OP** led him to beautiful results [Wils], which later led to the celebrated Askey-Wilson polynomials [AW]. Jim Wilson's independence was later wholeheartedly endorsed by Dick Askey, who said: "*If an authority in the field tells you that a certain approach is worth trying, listen to them. If they tell you that a certain approach is* **not** *worth trying,* **don't** *listen to them.*"

The 'uselessness' of (General Formula) was still proclaimed a few years later, by yet another authority, Jean Dieudonné, who said ([D], p. 11): "*La formule générale* [(General Formula)] *donnant les ... sont impraticables pour le calcul explicite ...*"

There is another way in which (General Formula) and its immediate corollaries **1** and **2** could be useful. Suppose that we know, *by other means*, that a certain set of explicitly given monic polynomials $Q_n(x)$ are orthogonal with respect to the functional T, i.e., $T(Q_n Q_m) = 0$ whenever $n \neq m$. Then by uniqueness, $Q_n = P_n$. If we are also able, using the explicit expression for $Q_n(x)$, to find $T(x^n Q_n(x))$, then Corollary **1** gives a way to *explicitly evaluate* the Hankel determinant Δ_n. If we are also able to explicitly compute $S(Q_n(x))$, then we would be able to evaluate the determinant Γ_n. This would be our strategy in the evaluation of the determinants on the left of (Not Yet Done), but we first need to digress again.

A Lean and Lively Course in q-Calculus

Until further notice,

$$(a)_n := (1-a)(1-qa)(1-q^2a)\dots(1-q^{n-1}a).$$

If I had my way, I would ban $1-$Calculus from the Freshman curriculum, and replace it by $q-$Calculus. Not only is it more fun, it also describes nature more accurately. The traditional calculus is based on the fictitious notion of the real line. It is now known that the universe is quantized, and if you are at point x, then the points that you can reach are in geometric progression $q^i x$, in accordance with Hubble expansion. The true value of q is almost, but *not quite* 1, and is a universal constant, yet to be determined.

The q-derivative, D_q, is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}.$$

The reader should verify that

$$D_q x^a = \frac{1-q^a}{1-q} x^{a-1},$$

and the *product rule*:

$$D_q[f(x) \cdot g(x)] = f(x) \cdot D_q g(x) + D_q f(x) \cdot g(qx). \quad \text{(Product Rule)}$$

The q-analog of integration, independently discovered by J. Thomae and the Rev. F. H. Jackson (see [AA], [GR]), is given by

$$\int_0^a f(x)\, d_q x := a(1-q) \sum_{r=0}^{\infty} f(aq^r) q^r,$$

and over a general interval:

$$\int_c^d f(x)\, d_q x := \int_0^d f(x)\, d_q x - \int_0^c f(x)\, d_q x.$$

The reader is invited to use telescoping to prove the following:

Fundamental Theorem of q-Calculus.

$$\int_c^d D_q F(x)\, d_q x \;=\; F(d) - F(c).$$

Combining the Product Rule and the Fundamental Theorem, we have:

q-Integration by Parts. *If $f(x)$ or $g(x)$ vanish at the endpoints c and d, then*

$$\int_c^d f(x) \cdot D_q g(x)\, d_q x = -\int_c^d D_q f(x) \cdot g(qx)\, d_q x.$$

Corollary. *If $g(q^i x)$ vanish at the endpoints c and d, for $i = 0, 1, \ldots, n-1$, then*

$$\int_c^d f(x) \cdot D_q^n g(x)\ d_q x = (-1)^n \int_c^d D_q^n f(x) \cdot g(q^n x)\ d_q x.$$

Even those who still believe in 1-Calculus can use q-Calculus to advantage. All they have to do is let $q \to 1$ at the end.

The q-analog of $\int_s^1 x^a dx = (1 - s^{a+1})/(a+1)$ is

$$\frac{1}{1-q} \int_s^1 x^a d_q x = \frac{1 - s^{a+1}}{1 - q^{a+1}}. \qquad \text{(q-Moment)}$$

Now this looks familiar! Letting $q := s^3$ in the definition of $N_{n+1}(X)$, given right after (Not Yet Done), (this new 'q' has nothing to do with the former Kuperberg q), we see that the matrix $N_{n+1}(1)$ is the Hankel matrix of the moments with respect to the functional

$$T(f(x)) := \frac{1}{1-q} \int_s^1 f(x) d_q x.$$

So all we need is to come up with orthogonal polynomials with respect to the 'q-Lebesgue-measure', over the interval $[s, 1]$.

q-Legendre Polynomials

The ordinary Legendre polynomials, over an interval (a, b) may be defined in terms of the Rodrigues formula

$$\frac{n!}{(2n)!} D^n \{(x-a)^n (x-b)^n\}.$$

The orthogonality follows immediately by integration by parts. This leads naturally to the q-analog,

$$Q_n(x; a, b) := \frac{(1-q)^n}{(q^{n+1})_n} D_q^n \{(x-a)(x-qa)\ldots(x-aq^{n-1}) \cdot (x-b)(x-qb)\ldots(x-bq^{n-1})\}.$$

Using q-integration by parts repeatedly, it follows immediately that the $Q_n(x; a, b)$ are orthogonal with respect to $q-$integration over (a, b). The classical case $a = -1$, $b = 1$ goes back to Markov. Askey and Andrews [AA] generalized these to q-Jacobi polynomials, and Askey and Wilson [AW] found the *ultimate* generalization. While at present I don't see how to apply these more general polynomials to combinatorial enumeration, I am sure that such a use will be found in the future, and all enumerators are urged to read [AA], [AW], and the modern classic [GR].

Going back to the determinant $N_{n+1}(X)$ of (Not Yet Done), we also need to introduce the functional, defined on monomials by:

$$S(x^j) = \frac{1 - X s^{n+j+1}}{1 - X^3 q^{n+j+1}},$$

and extended linearly.

Let $X := q^{\alpha/3}$. Then (recall that $s = q^{1/3}$):

$$S(x^j) = \frac{1 - s^{\alpha+n+j+1}}{1 - q^{n+j+\alpha+1}} = \frac{1}{1-q} \int_s^1 x^{\alpha+n} x^j \, d_q x.$$

By linearity, for any polynomial $p(x)$:

$$S(p(x)) = \frac{1}{1-q} \int_s^1 x^{\alpha+n} p(x) \, d_q x.$$

Using Corollary 2 of (General Formula), we get

$$\frac{\det N_{n+1}(X)}{\det N_{n+1}(1)} = \frac{\int_s^1 x^{\alpha+n} P_n(x) \, d_q x}{\int_s^1 x^n P_n(x) \, d_q x}, \qquad \text{(Almost Done)}$$

where $P_n(x)$ is now the $q-$Legendre polynomial over $[s, 1]$, $Q_n(x; s, 1)$ and $s = q^{1/3}$. In other words:

$$P_n(x) := \frac{(1-q)^n}{(q^{n+1})_n} D_q^n \{(x-1)(x-q) \dots (x - q^{n-1}) \cdot (x-s)(x-qs) \dots (x - sq^{n-1})\}.$$

Denouement

It remains to compute the right side of (Almost Done). Let's first do the denominator.

Proposition Bottom.

$$\frac{1}{1-q} \int_s^1 x^n P_n(x) \; d_q x = \frac{q^{n^2} (q)_n^2 (q^{-n} s)_{2n+1}}{(q^{n+1})_n (q^{n+1})_{n+1}}.$$

First Proof. Use q-integration by parts, n times (i.e., use the above corollary). The resulting q-integral is the famous q-Vandermonde-Chu sum, that evaluates to the right side. See [GR], or use `qEKHAD` accompanying [PWZ]. □

Remark. Proposition Bottom, combined with Corollary 1 of (General Formula) gives an alternative evaluation of Kuperberg's determinant $N_n(1)$, needed in [K].

Second Proof. Don't get off the shoulders of Greg Kuperberg yet. Use his evaluation, and Corollary 1 of (General Formula). □

Proposition Top. *Recalling that $X = q^{\alpha/3}$, we have*

$$\frac{1}{1-q} \int_s^1 x^{n+\alpha} P_n(x) \, d_q x = \frac{(-1)^n (qX^3)_n}{(q^{n+1})_n} \cdot \sum_{k=0}^{\infty} q^k X^{3k} \prod_{r=0}^{n-1} (q^{n+k} - q^r)(q^{n+k} - q^{r+1/3})$$

$$- \frac{(-1)^n (qX^3)_n}{(q^{n+1})_n} \cdot \sum_{k=0}^{\infty} q^{k+1/3} X^{3k+1} \prod_{r=0}^{n-1} (q^{n+k+1/3} - q^r)(q^{n+k+1/3} - q^{r+1/3}).$$

Proof. Let

$$F_n(x) := \frac{(1-q)^n}{(q^{n+1})_n}(x-1)(x-q)\dots(x-q^{n-1})\cdot(x-s)(x-qs)\dots(x-sq^{n-1}),$$

so that $P_n(x) = D_q^n F_n(x)$. Since $F_n(q^i x)$ vanish for $i = 0, \dots, n-1$ at both $x = 1$ and $x = s$, we have by q-integrating by parts n times (the above corollary), that

$$\begin{aligned}\int_s^1 x^{n+\alpha}\cdot P_n(x)\,d_qx &= \int_s^1 x^{n+\alpha}\cdot D_q^n F_n(x)\,d_qx\\ &= (-1)^n \int_s^1 D_q^n\{x^{n+\alpha}\}\cdot F_n(q^n x)\,d_qx\\ &= \frac{(-1)^n (q^{\alpha+1})_n}{(1-q)^n}\int_s^1 x^\alpha F_n(q^n x) d_qx.\end{aligned}$$

Now use the definition of q-integration over $[s, 1]$ and replace q^α by X^3, to complete the proof. □

To compute the right side of (Almost Done), we only need to divide the expression given by Proposition Top by the expression given by Proposition Bottom. Doing this, multiplying by $(1-s)^n = (1-q^{1/3})^n$, and taking the limit $q \to 1$, we get that the left side of (Not Yet Done) is the following expression. (Warning: Now we are safely back in 1−land, so from now $(a)_n := a(a+1)\dots(a+n-1)$, the ordinary rising factorial.)

$$\frac{(-1)^n(1-X^3)^n(2n+1)!}{3^n n!^3(-n+1/3)_{2n+1}}\left(\sum_{k=0}^{\infty}(k+1)_n(k+\frac{2}{3})_n X^{3k} - \sum_{k=0}^{\infty}(k+1)_n(k+\frac{4}{3})_n X^{3k+1}\right)$$

After trivial cancellations, equation (Not Yet Done) boils down to

$$\begin{aligned}&(\frac{1-X^3}{1-X})^{2n+1}\frac{(-1)^n(3n+1)!}{3^{n+1}n!^3(-n+1/3)_{2n+1}}\cdot\sum_{k=0}^{\infty}(k+1)_n(k+2/3)_n X^{3k}\\ &\quad-(\frac{1-X^3}{1-X})^{2n+1}\frac{(-1)^n(3n+1)!}{3^{n+1}n!^3(-n+1/3)_{2n+1}}\sum_{k=0}^{\infty}(k+1)_n(k+4/3)_n X^{3k+1}\\ &\quad= (\sqrt{-3})^n\sum_{r=0}^{n} w^{-r-n}\binom{n+r}{n}\binom{2n-r}{n}(1+wX)^r(1-w^2X)^{n-r}. \qquad \text{(Done)}\end{aligned}$$

This was given to `EKHAD`, the Maple package accompanying [PWZ]. `EKHAD` found a certain linear homogeneous second order recurrence in n that is satisfied by both sums on the left of (Done) (and hence by their difference), and also by the right side. It remains to prove that both sides of (Done) agree at $n = 0, 1$, which Maple did as well, even though it could be done by any human.

Links to the input and output files, and to `EKHAD` and `qEKHAD`, may be found at

`http://nyjm.albany.edu:8000/j/v2/Zeilberger-info.html`

The input and output files are called `inDone` and `outDone`, respectively.

Of course, your computer should be able to reproduce the file `outDone`: Once you have downloaded `EKHAD` and `inDone` into a directory, type:

```
maple -q < inDone > outDone
```

After 380 seconds of CPU time, `outDone` will be ready.

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX