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Lang's Conjectures, Fibered Powers, and Uniformity

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Abstract. We prove that the fibered power conjecture of Caporaso et al. (Conjecture H, [CHM], §6) together with Lang's conjecture implies the uniformity of rational points on varieties of general type, as predicted in [CHM]; a few applications on the arithmetic and geometry of curves are stated.

In an opposite direction, we give counterexamples to some analogous results in positive characteristic. We show that curves that change genus can have arbitrarily many rational points; and that curves over $\overline{\mathbb{F}_p}(t)$ can have arbitrarily many Frobenius orbits of non-constant points.

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1. Introduction

Let X be a variety of general type defined over a number field K. It was conjectured by S. Lang that the set of rational points $X(K)$ is not Zariski dense in X. In the paper [CHM] of L. Caporaso, J. Harris and B. Mazur it is shown that the above conjecture of Lang implies the existence of a uniform bound on the number of K-rational points of all curves of fixed genus g over K.

The paper [CHM] has immediately created a chasm among arithmetic geometers. This chasm, which sometimes runs right in the middle of the personalities involved, divides the loyal believers of Lang's conjecture, who marvel at this powerful implication, and the disbelievers, who try to use this implication to derive counterexamples to the conjecture.

In this paper we will attempt to deepen this chasm on both sides: first, using the techniques of [CHM] and continuing [ℵ], we prove more implications, some of which are very strong, of various conjectures of Lang. Along the way we will often use the *Fibered Power Conjecture*, also known as *Conjecture H* (see [CHM], §6) about higher dimensional varieties, which is regarded as very plausible among experts of higher dimensional algebraic geometry.

Second, we will show by way of counterexamples that two natural candidates for analogous statements in positive characteristic, are false.

Before we state any results, we need to specify various conjectures which we will apply.

1.1. A Few Conjectures of Lang. Let X be a variety of general type over a field K **of characteristic 0**. In view of Faltings's proof of Mordell's conjecture, Lang has stated the following conjectures:

Conjecture 1.1.

1. (Weak Lang conjecture) If K is finitely generated over $\mathbb{Q}$ then the set of rational points $X(K)$ is not Zariski dense in X.
2. (Weak Lang conjecture for function fields) If $k \subset K$ is a finitely generated regular extension in characteristic 0, and if $X(K)$ is Zariski dense in X, then X is birational to a variety X_0 defined over k and the *"non-constant points"* $X(K) \setminus X_0(k)$ are not Zariski dense in X.
3. (Geometric Lang conjecture) Assuming only $Char(K) = 0$, there is a proper Zariski closed subset $Z(X) \subset X$, called in [CHM] the *Langian exceptional set*, which is the union of all positive dimensional subvarieties which are not of general type.
4. (Strong Lang conjecture) If K is finitely generated over $\mathbb{Q}$ then there is a Zariski closed subset $Z \subset X$ such that for any finitely generated field $L \supset K$ we have that $X(L) \setminus Z(L)$ is finite.

These conjectures and the relationship between them are studied in [LangAMS], [LangIII] and in the introduction of [CHM]. For instance, it should be noted that the weak Lang conjecture together with the geometric conjecture imply the strong Lang conjecture. We remark that in the case of subvarieties of abelian varieties Lang's conjectures have been proven by Faltings ([Fal 92]).

1.2. The Fibered Power Conjecture. An important tool used by Caporaso et al. in [CHM] is that of fibered powers. Let $X \to B$ be a morphism of varieties in characteristic 0, where the general fiber is a variety of general type. We denote by X^n_B the n-th fibered power of X over B.

Conjecture 1.2. (The fibered power conjecture, or Conjecture H of [CHM]) For sufficiently large n, there exists a dominant rational map $h_n : X^n_B \dashrightarrow W_n$ where W_n is a variety of general type, and where the restriction of h_n to the general fiber $(X_b)^n$ is generically finite.

This conjecture is known for curves and surfaces:

Theorem FP 1. (Correlation Theorem of [CHM]) *The fibered power conjecture holds when $X \to B$ is a family of curves of genus > 1.*

Theorem FP 2. (Correlation Theorem of [Has]) *The fibered power conjecture holds when $X \to B$ is a family of surfaces of general type.*

Using their Theorem FP 1, and Lemma 1.1 of [CHM], Caporaso et al. have shown that the weak Lang conjecture implies a uniform bound on the number of rational points on curves (Uniform Bound Theorem, [CHM] Theorem 1.1).

Remark 1.3. It should be noted that the proofs of Theorems FP 1 and FP 2 give a bit more: they describe a natural dominant rational map $X^n_B \dashrightarrow W$. For the case of curves, if B_0 is the image of B in the moduli space $\mathbf{M}_g$, then for sufficiently large n the inverse image $B_n \subset \mathbf{M}_{g,n}$ in the moduli space of n-pointed curves is a variety of general type. Therefore the moduli map $X^n_B \dashrightarrow B_n$ satisfies the requirements. A similar construction works for surfaces of general type.

One may ask whether a description of this kind holds for higher dimensions.

It is convenient to make the following definitions when discussing Lang's conjectures:

Definition 1.4.
1. A variety X/K is said to be a **Lang variety** if there is a dominant rational map $X_{\overline{K}} \dashrightarrow W$, where W is a positive dimensional variety of general type.
2. A positive dimensional variety X is said to be **geometrically mordellic** (In short GeM) if $X_{\overline{K}}$ does not contain subvarieties which are not of general type.

In [LangIII], in the course of stating even more far reaching conjectures, Lang defined a notion of *algebraically hyperbolic* varieties, which is very similar, and conjecturally the same as that of GeM varieties. We chose to use a different terminology here, to avoid confusion.

Note that the weak Lang conjecture directly implies that the rational points on a Lang variety over a number field are not Zariski dense, and that there are only finitely many rational points over a number field on a GeM variety.

1.3. Summary of Results on the Implication Side. As indicated in [CHM] §6, the fibered power conjecture together with Lang's conjectures should have very strong implications for counting rational points on varieties of general type, similar to the Uniform Bound Theorem of [CHM]. Here we will prove the following basic result:

Theorem 1.5. *Assume that the weak Lang conjecture as well as the fibered power conjecture hold. Let $X \to B$ be a family of GeM varieties over a number field K (or any finitely generated field over Q). Then there is a uniform bound on $\sharp X_b(K)$.*

One may refine this theorem for arbitrary families of varieties of general type, obtaining a bound on the number of points which do not lie in Langian exceptional sets of the fibers. If one assumes the geometric Lang conjecture, one obtains a closed subset $Z(X_b)$ for every $b \in B$. A natural question which arises in such a refinement is: how do these subsets fit together? An answer was given in [CHM], Theorem 6.1, assuming the fibered power conjecture as well: the varieties $Z(X)$ are uniformly bounded. We will show that, using results of Viehweg, one does not need to assume the fibered power conjecture:

Theorem 1.6. (Compare [CHM], Theorem 6.1.) *Assume that the geometric Lang conjecture holds. Let $X \to B$ be a family of varieties of general type. Then there is a proper closed subvariety $\tilde{Z} \subset X$ such that for any $b \in B$ we have $Z(X_b) \subset \tilde{Z}$.*

Using Theorem 1.6, we can apply Theorem 1.5 to any family $X \to B$ of varieties of general type, assuming that the geometric Lang conjecture holds: we can bound the rational points in the complement of $\tilde{Z}$.

We will apply our Theorem 1.5 in various natural cases. An immediate but rather surprising application is the following theorem:

Theorem 1.7. *Assume that the weak Lang conjecture as well as the fibered power conjecture hold. Let $X \to B$ be a family of GeM varieties over a field K finitely generated over $\mathbb{Q}$. Fix a number d. Then there is an integer N_d such that for any field extension L of K of degree d and every $b \in B(L)$ we have $\sharp X_b(L) < N_d$.*

As a corollary, we see that Lang's conjecture together with fibered power conjecture imply the existence of a bound on the number of points on curves of fixed genus g over a number field L which depends only on the degree of the number field $[L : \mathbb{Q}]$.

These results have natural analogues for function fields. We will state a few of these, notably:

Theorem 1.8. *Assume that Lang's conjecture for function fields holds. Fix an integer $g > 1$. Then there is an integer $N(g)$ such that for any generically smooth fibration of curves $C \to D$ where the fiber has genus g and the base D is a* **hyperelliptic** *curve, there are at most N non-constant sections $s : D \to C$.*

We remind the reader that the *gonality* of a curve D is the minimal degree of a nonconstant rational function on D (so a curve of gonality 2 is hyperelliptic). One expects the above theorem to be generalized to the situation where "hyperelliptic curve" is replaced by "curve of gonality $\leq d$" for fixed d.

Historical remark 1.9. The idea that the number of solutions of members of a family of diophantine equations should be uniformly bounded, when finite, goes back to Siegel (see [Siegel], §II.7, page 262). The reasoning seems based on the naive idea of eliminating coefficients (see e.g. [Chowla]). This idea, coupled with the generalized *abc* conjecture can be made to work in some cases, for function fields of characteristic zero (see [Mueller], [Bo-Muel]). Lapin (see [Lapin] and references there) has proposed an argument suggesting that uniform bounds should fail over

$\mathbb{C}(t)$ (contradicting the geometric Lang conjecture, and in particular Theorem 1.8), but we have been informed in private communications that there are gaps in the arguments there.

1.4. Summary of Results: Examples in Positive Characteristic. It has been a long tradition to test the plausibility of conjectures in arithmetic geometry by finding analogous results for function fields in positive characteristic. Our results here are negative: two natural analogues of the Uniform Bound Theorem and of our results in characteristic 0 are false.

One may try to transpose Lang's conjectures to the case of positive characteristic, but they are trivially false already in the case of curves. Two natural approaches to restore the conjecture, which work for curves, are either to insist on non - isotriviality of the variety or to look at points which are not in the image of the Frobenius map. A general statement for subvarieties of abelian varieties was studied in [A-V] and completed in [Hru]. Unfortunately both these approaches fail already for surfaces. In fact, there are unirational surfaces of general type in positive characteristic, and even non-constant families of those, which provide counterexamples to such conjectures.

One may try to look at varieties with non-zero Kodaira - Spencer class, which is a condition slightly stronger than non - isotriviality, but there seem to be counterexamples here as well. Again the problem is due to unirational varieties. In all these examples the surfaces have a large set of birational endomorphisms (coming either from the Frobenius or from birational endomorphisms of $\mathbb{P}^2$), and one may try to take these into account in stating a Lang type conjecture. A rather drastic approach is to look only at varieties which are not covered by non-general type varieties, but this would be an unsatisfactory and almost unverifiable conjecture due to the fact that it not known how to tell whether a variety of general type can be covered by a variety which is not of general type. See some related discussion in [Vol-surv].

One may still ask, to what extent the statments which are implied by Lang's conjecture in characteristic 0 can be transposed to positive characteristics. Here we will address the question of uniformity of rational points on curves.

By a classical result of Samuel [Samuel], if K is a function field in characteristic $p > 0$ and C is a non-isotrivial curve of genus > 1, then $C(K)$ is finite; and if C is isotrivial, then there are only finitely many points which are not defined over the field of p-th powers K^p.

Supposing C as above is a smooth curve with non-zero Kodaira-Spencer class, it is not known if one can obtain a uniform bound on the number of rational points $C(K)$ (but see [Bu-Vol] for a strong bound on $\sharp C(K)$ depending on the Mordell-Weil rank of $J(C)$). In our first example, we will consider the case of non-smooth curves, or *curves that change genus* (see discussion in 4.1). It was shown in [Vol-91], analogously to Samuel's Theorem, that a curve C that changes genus has a finite set of rational points. We will show (Theorem 4.1) that such curves may have arbitrarily many rational points.

The second example is concerned with isotrivial curves. We work over the function field $K = \overline{\mathbb{F}_p}(t)$, and construct isotrivial curves C with arbitrarily many rational points $C(K)$ which are not in $C(K^p)$. In particular this implies that Proposition 3.5

and Corollary 3.7 below may have no satisfactory analogues in positive characteristic.

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2. Proof of Theorem 1.5

2.1. Preliminaries. Throughout subsection 2.1 **we assume that the fibered power conjecture holds**, and the base field is algebraically closed.

Observe that a positive dimensional subvariety of a GeM variety is GeM; and the normalization of an GeM variety is GeM. Note also that a variety dominating a Lang variety is a Lang variety as well.

Proposition 2.1. *Let $X \to B$ be a family of GeM varieties. Let $F \subset X$ be a reduced subscheme such that every component of F dominating B has positive fiber dimension. Then for n sufficiently large, every component of the fibered power F^n_B which dominates B is a Lang variety.*

The proof will use the following lemmas:

Lemma 2.2. *Let $X \to B$ and F be as above, and assume that the general fiber of $F \to B$ is irreducible. Then for n sufficiently large, the dominant component of F^n_B is a Lang variety.*

Proof. Apply the fibered power conjecture to $F \to B$, using the fact that the fibers of F are of general type. □

Lemma 2.3. *Let $X \to B$ and F be as in the proposition, with F irreducible. Then for n sufficiently large, every component of the fibered power F^n_B which dominates B is a Lang variety.*

Proof. Let $\tilde{F}$ be the normalization of F, and let $\tilde{F} \to \tilde{B} \to B$ be the Stein factorization. Denote by c the degree of $\tilde{B}$ over B. Let $G \subset \tilde{F}^n_B$ be a dominant component. Then G parametrizes n-tuples of points in the fibers of $\tilde{F}$ over B, and since G is irreducible, there is a decomposition $\{1, \dots, n\} = \cup_{i=1}^c J_i$ and G surjects onto the dominant component of $\tilde{F}^{J_i}_{\tilde{B}}$. At least one of the subsets J_i has at least n/c elements. Using Lemma 2.2 applied to $\tilde{F} \to \tilde{B}$, we see that for n/c large enough, G is a Lang variety. □

Proof of Proposition 2.1. Let $F = F_1 \cup \ldots \cup F_m$ be the decomposition into irreducible components. Let G be a dominant component of F^n_B. Then G dominates $(F_1)^{n_1}_B \times_B \cdots \times_B (F_m)^{n_m}_B$. For at least one i we have $n_i > n/m$, so applying the previous lemma we obtain that G is a Lang variety. □

2.2. Prolongable Points. We return to the setup in Theorem 1.5.

Definition 2.4.
1. A point $x_n = (P_1, \dots, P_n) \in X^n_B(K)$ is said to be *off-diagonal* if for any $1 \le i < j \le n$ we have $P_i \neq P_j$. We extend this for $n = 0$ trivially by agreeing that any point of $B(K)$ is off-diagonal.
2. Let $m > n$. An off-diagonal point x_n is said to be *m-prolongable* if there is an off-diagonal "prolongation" $x_m \in X^m_B(K)$ whose first n coordinates give x_n.

Let $E_n^{(m)}$ be the set of m-prolongable points on X_B^n, and let $F_n^{(m)}$ be the Zariski closure. Let $F_n = \cap_{m>n} F_n^{(m)}$. By the Noetherian property of the Zariski topology we have $F_n = F_n^{(m)}$ for some m.

If some fiber of $X \to B$ contains a large number m of points, then $E_n^{(m)}$ is nonempty. If the number m is not bounded, then F_n is nonempty. Therefore, in order to bound the number of rational points on each fiber, all we need to show is $F_n = \emptyset$ for some n.

Lemma 2.5. *We have a surjection $F_{n+1} \to F_n$.*

Proof. The set $E_{n+1}^{(m)}$ surjects to $E_n^{(m)}$ for any $m > n+1$. □

Lemma 2.6. *Every fiber of $F_{n+1} \to F_n$ is positive dimensional.*

Proof. Suppose that over an open set in F_n the degree of the map is d. Then $E_n^{(n+d+1)}$ cannot be dense in F_n: if $(y_1, \ldots, y_{n+d+1})$ is an off-diagonal prolongation of $(y_1, \ldots, y_n) \in E_n^{(n+d+1)}$, then for $n+1 \leq j \leq n+d+1$ we have that the points $(y_1, \ldots, y_n, y_j) \in E_{n+1}^{(n+d+1)}$ are $d+1$ distinct points lying in a fiber of $F_{n+1} \to F_n$, therefore the degree of the map is at least $d+1$. □

2.3. Proof of Theorem 1.5. Denote by r the fiber dimension of $X \to B$. We show by induction on i, that for any n and i, the dimension of any fiber of $F_{n+1} \to F_n$ is at least $i+1$. This will lead to a contradiction, since by definition the fiber dimension of $F_{n+1} \to F_n$ is at most r. Lemma 2.6 shows this for $i = 0$. Assume it holds true for $i-1$, let $n \geq 0$ and let G be a component of F_n, such that the fiber dimension of F_{n+1} over G is i. Applying the inductive assumption to each $F_{n+j+1} \to F_{n+j}$, we have that the dimension of every fiber of F_{n+k} over F_n is at least ik. On the other hand, by definition F_{n+k} is a subscheme of the fibered power $(F_{n+1})^k_{F_n}$, so over G it has fiber dimension precisely ik. Therefore there exists a component H_k of F_{n+k} dominant over G of fiber dimension ik, which is therefore identified as a dominant component of the fibered power $(F_{n+1})^k_{F_n}$. By Proposition 2.1, for k sufficiently large we have that H_k is a Lang variety. Lang's conjecture implies that $H_k(K)$ is not dense in K, contradicting the definition of F_{n+k}. □

Remark 2.7. Note that in the proof we have applied the fibered power conjecture for families of fiber dimension i, where i is at most the fiber dimension of the family $X \to B$. Therefore in case the fibers of $X \to B$ are curves or surfaces, one may apply Theorems FP 1 and FP 2 instead of the fibered power conjecture.

3. A Few Refinements and Applications in Arithmetic and Geometry

3.1. Proof of Theorem 1.6. Assume that $X \to B$ is a family of varieties of general type. By Hironaka's desingularization theorem, we may assume that B is a smooth projective variety. Let L be a very ample line bundle on B, such that $K_B^{\otimes 2} \otimes L$ is ample as well. Let H be a smooth divisor associated to a section of $L^{\otimes 2}$. Let $\pi : B_1 \to B$ be the cyclic double cover ramified along H. By adjunction, B_1 is a variety of general type: $K_{B_1}^{\otimes 2} \simeq \pi^*(K_B^{\otimes 2} \otimes L)$. Let $X_1 \to X$ be the pullback of X to B_1. By the main theorem (Satz III) of [Vie], the variety X_1 is of general type.

Assuming the geometric Lang conjecture, Let $Z_1(X_1)$ be the Langian exceptional set. Let $\tilde{Z}$ be the image of $Z_1(X_1)$ in X. Then for any $b \in B$, we have by definition that $Z(X_b) \subset \tilde{Z}$. □

It has been noted in [CHM] that Viehweg's work goes a long way towards proving the fibered power conjecture. It is therefore not surprising that it may be used on occasion to replace the assumption of fibered power conjecture.

3.2. Uniformity in Terms of the Degree of an Extension. Let $X \to B$ be a family of GeM varieties over K. Assuming the conjectures, Theorem 1.5 gave us a uniform bound on the number of rational points over finite extension fields in the fibers. We will now see that this in fact implies a much stronger result, namely our Theorem 1.7: the uniform bound only depends on the degree of the field extension.

Proof of Theorem 1.7. For $n = 1$ or 2, Let $Y_n = \operatorname{Sym}^d(X^n_B)$, and $Y_0 = \operatorname{Sym}^d(B)$. We have natural maps $p_n : Y_n \to Y_{n-1}$. Let Γ be the branch locus of the quotient map $X^d \to Y_1$, namely the set of points which are fixed by some permutation. If $P \notin \Gamma$ then $p_2^{-1}(P)$ is a GeM variety, isomorphic over $\overline{K}$ to the product of d fibers of the family $X \to B$. Denote $Y_1' = Y_1 \setminus \Gamma_1$, and $Y_2' = p_2^{-1} Y_1'$. Then $Y_2' \to Y_1'$ is a family of GeM varieties, and by Theorem 1.5 we have a bound on the cardinality of $p_2^{-1}(y)(K)$ uniformly over all $y \in Y_1'(K)$.

By induction, it suffices to bound the number of points in $X_b(L)$ over any field L of degree d over K, which are defined over L but not over any intermediate field. If $\sigma_1, \ldots, \sigma_d$ are the distinct embeddings of L in $\overline{K}$, and $P \in X_b(L)$ not defined over any intermediate field, then the points $\sigma_i(P) \in X_{\sigma_i(b)}(\sigma_i(K)) \subset X(\overline{K})$ are distinct. If $(P_1, P_2) \in X^2_B(L)$ is a pair of such points, then the Galois orbit $\{\sigma_i(P_1, P_2), i = 1, \ldots, d\}$ is Galois stable, therefore it gives rise to a point in $Y_2(K)$. This point has the further property that its image in Y_1 does not lie in Γ_1, so it is in $Y_2'(K)$. The previous paragraph shows that the number of points on a fiber is bounded. □

Applying Theorem 1.7 where $X \to B$ is the universal family over the Hilbert scheme of 3-canonical curves of genus g (as in [CHM], Subsection 1.2), we obtain the following:

Corollary 3.1. *Assume that the weak Lang conjecture as well as the fibered power conjecture hold. Fix integers $d, g > 1$ and a number field K. Then there is a uniform bound N_d such that for any field extension L of K of degree d and every curve C of genus g over L we have $\sharp C(L) < N_d$.*

Remark 3.2. In the cases of degrees $d \leq 3$ one does not need to assume the fibered power conjecture: this was proven in [ℵ], using the fact that the fibered power conjecture holds for families of curves or surfaces. A similar result has been recently announced by P. Pacelli for arbitrary d.

Here is a special case: let $f(x) \in \mathbb{Q}(x)$ be a polynomial of degree > 4 with distinct complex roots. Then, assuming the weak Lang conjecture, the number of rational points over any quadratic field on the curve $C : y^2 = f(x)$ is bounded uniformly. We remark that, if $\deg f > 6$, this in fact may be deduced using a combination of [CHM] and the following theorem of Vojta [Voj]: all but finitely many quadratic points on C have rational x coordinate. One then applies [CHM] which gives a uniform bound on the rational points on the twists $ty^2 = f(x)$.

Following the suggestion of [CHM], §6 one can apply Theorem 1.5 to symmetric powers of curves. Since the fibered power conjecture is known for surfaces, one obtains the following (stated without proof in [CHM], Theorem 6.2):

Corollary 3.3. (Compare [CHM], Theorem 6.2) *Assume that the weak Lang conjecture holds. Fix a number field K. Then there is a uniform bound N for the number of quadratic points on any non-hyperelliptic, non-bielliptic curve C of genus g over K.*

Similarly, it was shown in [A-H], Lemma 1 that if the gonality of a curve C is $> 2d$ then $\mathrm{Sym}^d(C)$ is GeM, being a subvariety of an abelian variety not containing translated abelian subvarieties. Recall that a closed point P on C is said to be of degree d over K if $[K(P):K] = d$. We deduce the following:

Corollary 3.4. *Assume that the weak Lang conjecture holds. Fix a number field K and an integer d. Then there is a uniform bound N for the number of points of degree d over K on any curve C of genus g and gonality $> 2d$ over K.*

3.3. The Geometric Case. One can use the same methods using Lang's conjecture for function fields of characteristic 0, say over $\mathbb{C}$. Given a fibration $X \to B$ where the generic fiber is a variety of general type, a rational point $s \in X(K_B)$ over the function field of B is called *constant* if X is birational to a product $X_0 \times B$ and s corresponds to a point on X_0. Lang's conjecture for function fields says that the non-constant points are not Zariski dense.

In this section we will restrict attention to the case where the base is the projective line $\mathbb{P}^1$. We will only assume the following statement: if X is a variety of general type, then the rational curves in X are not Zariski dense. It is easy to see that this statement in fact follows from the geometric Lang conjecture, as well as from Lang's conjecture for function fields.

We would like to apply this conjecture to obtain geometric uniformity results. One has to be careful here, since the geometric Lang conjecture cannot be applied to Lang varieties, and one has to use a variety of general type directly.

As stated in the introduction, if $X \to B$ is a family of curves of genus > 1 the appropriate variety W of general type dominated by X^n_B is the image $B_n \subset \mathbf{M}_{g,n}$ of the moduli map $X^n_B \dashrightarrow \mathbf{M}_{g,n}$. This is used in the proof of the following proposition:

Proposition 3.5. *Assume that Lang's conjecture for function fields holds. Fix an integer $g > 1$. Then there is an integer N such that for any generically smooth family of curves $C \to \mathbb{P}^1$ of genus g there are at most N non-constant sections $s : \mathbb{P}^1 \to C$.*

Proof. First note that if $s : \mathbb{P}^1 \to C$ is a nonconstant section whose image in $\mathbf{M}_{g,1}$ is a point, then s becomes a constant section after a finite base change $D \to \mathbb{P}^1$. This implies that s is fixed by a nontrivial automorphism of C, and the number of such points is bounded uniformly in terms of g. Therefore it suffices to bound the number of sections whose image in $\mathbf{M}_{g,1}$ is non-constant. We will call such sections *strictly non-constant.*

Let $B_0 \subset \mathbf{M}_g$ be a closed subvariety, and choose n such that $B_n \subset \mathbf{M}_{g,n}$ is of general type. If a family $C \to \mathbb{P}^1$ has moduli in B_0, then for any n-tuple of strictly non-constant sections $s_i : \mathbb{P}^1 \to C \quad i = 1, \leq n$, we obtain a non-constant rational map $\mathbb{P}^1 \to B_n$. Let $F \subset B_n$ be the Zariski closure of the images of the collection of non-constant rational maps obtained this way.

Since B_n is of general type, Lang's conjecture implies that $F \neq B_n$. Applying Lemma 1.1 of [CHM] we obtain that there is an closed subset $F_0 \subset B_0$ and an integer N such that, given a family of curve $C \to \mathbb{P}^1$ such that the rational image of $\mathbb{P}^1$ in $\mathbf{M}_g$ lies in B_0 but not in F_0, there are at most N strictly non-constant sections of C. Noetherian induction gives the theorem. □

Choosing a coordinate t on $\mathbb{P}^1$ we can pull back the curve C along the map $\mathbb{P}^1 \to \mathbb{P}^1$ obtained by taking n-th roots of t. Let $\mathbb{C}(t^{1/\infty}) = \mathbb{C}(\{t^{1/n}, n \geq 1\})$, the field obtained by adjoining all roots of t. If one restricts attention to non-isotrivial curves, one obtains the following amusing result:

Corollary 3.6. *Assume that Lang's conjecture for function fields holds. Fix an integer $g > 1$. Then there is an integer N such that for any smooth nonisotrivial curve C over $\mathbb{C}(t)$ of genus g there are at most N points in $C(\mathbb{C}(t^{1/\infty}))$.*

One can also try to prove uniformity results analogous to Theorem 1.7. Using the results in [ℵ] we can refine Proposition 3.5 and obtain Theorem 1.8.

Proof of Theorem 1.8. The proof is a slight modification of the theorem of [ℵ], keeping track of the dominant map to a variety of general type.

As in the proof of Theorem 1.7, it suffices to look at sections $s : D \to C$ which are not pullbacks of sections of families over $\mathbb{P}^1$.

In an analogous way to the proof of Theorem 1.5, we say that an n-tuple of distinct, strictly non-constant sections is m-prolongable if it may be prolonged to an m-tuple of distinct, strictly non-constant sections, none of which being the pullback from a family over $\mathbb{P}^1$. Any n-tuple of distinct sections $s_i : D \to C$ over a hyperelliptic curve D gives rise to a rational map $\mathbb{P}^1 \to \mathrm{Sym}^2(\mathbf{M}_{g,n})$. We define $F_n^{(m)}$ to be the closure in $\mathrm{Sym}^2(\mathbf{M}_{g,n})$ of the images of m-prolongable sections, and $F_n = \cap_{m>n} F_n^{(m)}$.

As in Lemma 2.6, we have that the relative dimension of any fiber of $F_{n+1} \to F_n$ is positive. We have two cases to consider: either for high n there is a component of F_{n+1} having fiber dimension 1 over F_n, or for all n the fiber dimension is everywhere 2.

In case the fiber dimension is 1, we will see that there is a component of F_{n+k} which is a variety of general type. Assuming Lang's conjecture for function fields this contradicts the fact that the images of non-constant sections are dense. Fix a general fiber f of F_{n+1} over F_n. The curve f lies inside a surface isomorphic to the product of two curves $C_{b_1} \times C_{b_2}$. By the definition of m-prolongable sections, and an argument identical to that of Lemma 2.6, we obtain that there is a component f' of f which maps surjectively to both C_{b_1} and C_{b_2}. Therefore as the pair $\{b_1, b_2\}$ moves in $\mathrm{Sym}^2(\mathbf{M}_g)$, the curve f' moves in moduli as well.

Let F' be a component of F_{n+1} whose fibers have the above property, namely they surject to both factors C_{b_1} and C_{b_2}. Let F'_k be any component of $(F')^k_{F_n}$. Following the proof of Proposition 2.1 one easily sees that it suffices to show that for large k, F'_k is of general type.

If we use the moduli description of the dominant map to a variety of general type $m : F'_k \dashrightarrow W$ constructed through Proposition 2.1, we see that if E is a general curve in F'_k lying in a fiber of m, then E projects to a point in $\mathrm{Sym}^2(\mathbf{M}_g)$; and moreover, E projects to an off-diagonal point in F_l for some $1 \leq l \leq n+k$. But the fibers over off-diagonal points are GeM varieties, therefore the general fiber of the map m is of general type. By the main theorem of [Vie], F'_k is itself a variety of general type.

In case the map $F_{n+1} \to F_n$ has fiber dimension 2, we use Proposition 1 of [ℵ]: let $B' \subset \mathrm{Sym}^2(\mathbf{M}_g)$. Then for high n, the inverse image $B'_n \subset \mathrm{Sym}^2(\mathbf{M}_{g,n})$ of B' is a variety of general type. Since the images of non-constant sections are dense in F_n, this again contradicts Lang's conjecture. □

If one restricts attentions to trivial fibrations, one obtains as an immediate corollary:

Corollary 3.7. *Assume that the Lang conjecture for function fields holds. Fix an integer $g > 1$. Then there is an integer N such that for any curve C over $\mathbb{C}$ of genus g and any hyperelliptic curve D there are at most N non-constant morphisms $f : D \to C$.*

It should be noted that the theory of Hilbert schemes gives the existence of a bound depending on the genus of D, which is however not as strong. As in the arithmetic case, P. Pacelli has recently announced a generalization of these results to the case where D is d-gonal, for fixed d.

In the special case where one considers maps induced by automophisms, a version of the corollary above can be proven without assuming Lang's conjecture:

Proposition 3.8. *There is an integer $N(g)$, such that if D is a hyperelliptic curve in characteristic 0, $G = Aut(D)$, $H < G$ a subgroup and $C = D/H$ is a curve of genus $g > 1$, then $[G : H] \leq N(g)$.*

Remark 3.9. We will show later that this proposition fails in positive characteristic.

Proof. We have a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & C \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{f'} & \mathbb{P}^1 \end{array}$$

Since $g > 0$, we have an embedding $H \subset Aut(\mathbb{P}^1)$. By the Riemann - Hurwitz formula we have

$$2g(D) - 2 = |H|(2g-2) + r,$$

and on the other hand $2|H| - 2 = r'$, where r' is the degree of the ramification divisor of f'. Clearly $r \leq 2r'$, therefore

$$2g(D) - 2 \leq (2g+2)|H|.$$

Since $|G| < 84(g(D)-1)$ we get $[G : H] \leq |G|(2g+2)/(2g(D)-2) \leq 42(2g+2)$. □

4. Examples in Positive Characteristic

4.1. Curves that Change Genus Can Have Arbitrarily Many Rational Points. Let K be a global field of positive characteristic p. In other words, K is a function field in one variable over a finite field of characteristic p. Let C be a projective algebraic curve defined over K. One defines the absolute genus of C, in the usual way, by extending the field to the algebraic closure. We also define the genus of C relative to K to be the integer g_K that makes the Riemann-Roch formula hold, that is, for any K-divisor D of C, of sufficiently large degree, the dimension, $l(D)$, of the K-vector space of functions of $K(C)$ whose polar divisor is bounded by D, is $\deg D + 1 - g_K$. Since K is not perfect, the relative genus may change under inseparable extensions. (See e.g., [Artin] or [Tate]). It was shown in [Vol-91] that if the genus of C relative to K is different from the absolute genus of C then $C(K)$ is finite. The proof in [Vol-91] can be easily adapted to give an upper bound for $\sharp C(K)$, which however depends on C. In this section we give examples of curves C/K with fixed g_K for which $\sharp C(K)$ is arbitrarily large.

Theorem 4.1. *Let $p > 2$ be a prime and $q = p^n$. Consider the curve $C_n/\mathbb{F}_p(t)$ defined by*

$$x - (t + t^{q+2} + t^{2q+3} + \cdots + t^{(p-2)q+p-1})x^p = y^p.$$

The curve C_n has absolute genus zero but has genus relative to $\mathbb{F}_p(t)$ equal to $(p-1)(p-2)/2$. Furthermore $\sharp C_n(\mathbb{F}_p(t)) \geq p^{2^n/2n}$ and $\sharp C_n(\mathbb{F}_{p^{2n}}(t)) \geq p^{2^n}$.

Proof. We will construct points on C_n whose x-coordinate is of the form

$$a(t)/(t^{q+1} - 1),$$

where $a(t) = \sum_{i=0}^{q-1} \alpha_i t^i$. We will get a point in $C_n/\overline{\mathbb{F}_p}(t)$ if

$$(t^{q+1} - 1)^{p-1} a(t) - (t + t^{q+2} + t^{2q+3} + ... + t^{(p-2)q+p-1})a(t)^p$$

is a p-th power. Using the fact that $(t^{q+1} - 1)^{p-1} = \sum_{i=0}^{p-1} t^{(q+1)i}$ and comparing coefficients, this condition is equivalent to:

$$\alpha_i = \begin{cases} \alpha_{(i+q)/p}^p & i \equiv 0 \pmod p \\ \\ \alpha_{(i-1)/p}^p & i \equiv 1 \pmod p \\ \\ \alpha_i = 0 & \text{otherwise.} \end{cases}$$

Consider the map $\phi(i)$ defined for positive integers i, $\quad i \equiv 0, 1 \pmod p$ by

$$\phi(i) = \begin{cases} (i+q)/p & i \equiv 0 \pmod p \\ (i-1)/p & i \equiv 1 \pmod p \end{cases}$$

It has the following alternate description for $i < q$. If $i = \sum_{j=0}^{n-1} \epsilon_j p^j$, $0 \leq \epsilon_j \leq p-1$, $\phi(i) = \sum_{j=1}^{n-1} \epsilon_j p^{j-1} + \delta p^{n-1}$, where $\delta = 1$ if $\epsilon_0 = 0$ and $\delta = 0$ if $\epsilon_0 = 1$. In other words, the digits in base p, $(\epsilon_{n-1}, \ldots, \epsilon_0)$ are replaced by $(1 - \epsilon_0, \epsilon_{n-1}, \ldots, \epsilon_1)$. It follows that if $\epsilon_j \neq 0, 1$ for some j, then $\phi^r(i) \not\equiv 0, 1 \pmod p$ for some $r > 0$. On the other hand, if $\epsilon_j = 0, 1$ for all j then $\phi^r(i)$ is defined for all $r > 0$. Moreover it is easy to check that, in this case, $\phi^{2n}(i) = i$.

Returning to our α_i's, we see that $\alpha_i = 0$ if $\epsilon_j \neq 0, 1$ for some j and that $\alpha_{\phi(i)}^p = \alpha_i$ and $\alpha_i^{p^{2n}} = \alpha_i$ if $\epsilon_j = 0, 1$ for all j. If $\alpha_i \in \mathbb{F}_p$ this simply means $\alpha_{\phi(i)} = \alpha_i$. The set of polynomials $a(t) \in \mathbb{F}_p[t]$ satisfying our conditions form an $\mathbb{F}_p$-vector space and each orbit of ϕ contributes one dimension to it. Since each orbit has at most $2n$ elements and there are 2^n distinct $i = \sum_{j=0}^{n-1} \epsilon_j p^j$, $\epsilon_j = 0, 1$, we obtain at least $2^n/2n$ orbits, hence the count for $\mathbb{F}_p(t)$. In the case of $\mathbb{F}_{p^{2n}}(t)$, an orbit of length r contributes an r-dimensional $\mathbb{F}_p$-vector space. Since $r|2n$, the theorem follows. □

Remark 4.2. It can be shown, using the methods of [Vol-91], that indeed the points produced in the proof of the theorem are all the rational points of C_n.

Remark 4.3. In the case $p = 3$, C_n is a quasi-elliptic fibration over $\mathbb{P}^1$ in the sense of the classification of surfaces ([Bo-Mum], [L]) and our result shows that the 3-rank of the group of sections (the "Mordell-Weil" group) can be arbitrarily large. This was also shown by Ito [Ito].

Remark 4.4. The curves C_n are the members of the family of curves $x - tf(u)x^p = y^p$, where $f(u) = \sum_{i=0}^{p-2} u^i$, for $u = t^{p^n+1}$. It follows from the results of [Vol-91] that $tf(u)$ is a p-th power in $\mathbb{F}_p(t)$ for only finitely many $u \in \mathbb{F}_p(t)$, so the curve corresponding to a given $u \in \mathbb{F}_p(t)$ has finitely many points for all but finitely many u's, again by the results of [Vol-91]. Following [CHM] we consider the total space of the family, that is, the surface S over $\mathbb{F}_p(t)$ defined by $x - tf(u)x^p = y^p$ and, as is shown in [CHM], the set of rational points of S is Zariski dense, for otherwise, the theorem above would be violated. Since S is unirational, it is not surprising that this holds for some extension of $\mathbb{F}_p(t)$, but since S cannot be covered by $\mathbb{P}^2$ over $\mathbb{F}_p(t)$, it is surprising that this occurs over $\mathbb{F}_p(t)$. Also, S is of general type for $p \geq 7$, thus showing that Lang's conjecture cannot be easily transposed to positive characteristic.

4.2. Hyperelliptic Curves Over $\overline{\mathbb{F}_p}(t)$ Can Have Arbitrarily Many Frobenius - Orbits of Nonconstant Points. Let $p > 3$ be a prime, and let $q = p^n$. Let C_n be the curve over $K = \overline{\mathbb{F}_p}(t)$ defined by the equation

$$y^2 = (x^p - x)(t^q - t).$$

The curve C_n is hyperelliptic of genus $(p-1)/2 > 1$. For each $b \in \mathbb{F}_q^\times$ let $a = b^2$, and define

$$x_b(t) = \sum_{i=0}^{n-1} (at)^{p^i}; \quad y_b(t) = b(t^q - t).$$

Since $a^q = a$ we have that $x_b(t)^p - x_b(t) = at^q - at = a(t^q - t)$, therefore $y_b(t)^2 = (x_b(t)^p - x_b(t))(t^q - t)$, namely $(x_b(t), y_b(t)) \in C_n(K)$. We thus obtained $q - 1$ different non-constant points on C_n, and since none of them is defined over K^p, they belong to different Frobenius orbits.

This example can be used to show at the same time that Corollary 3.7 fails in positive characteristic. For let C be the curve $y^2 = x^p - x$ and let D_n be the curve $u^2 = t^q - t$. Then for any $b \in \mathbb{F}_q^\times$ we obtain a separable morphism $f_b : D \to C$ via $x = \sum_{i=0}^{n-1} (at)^{p^i}; \quad y = bu$.

Notice that one has $f_b = f_1 \circ \sigma_b$ where $\sigma_b : D \to D$ is the automorphism $(t, u) \to (b^2 t, bu)$. Moreover, C is the quotient of D under the action a group H as follows:

$$H = \{\tau \in Aut(D) | \tau(u, v) = (u + c, v) \text{ for some } c \in \mathbb{F}_q, c + c^p + \cdots + c^{p^{n-1}} = 0\}.$$

Therefore this example contradicts Proposition 3.8 above as well.

Another geometric phenomenon arising from this example is the following: let C be an isotrivial hyperelliptic curve given by $y^2 = f(x)$. Let $i : C \to C$ be the hyperelliptic involution. Let $S = C \times C/i \times i$ be the quotient by the involution acting diagonally, which is a surface of general type. We can describe S using the equation $z^2 = f(x_1)f(x_2)$. There are many rational curves on the surface S: for instance, the diagonal $x_2 = x_1, z = f(x_1)$, and the "graphs of Frobenius" $x_2 = x_1^q$, $z = f(x_1)^{(q+1)/2}$. Notice that these curves lie in the same orbit of the inseparable birational endomorphism

$$(x_1, x_2, z) \mapsto (x_1, x_2^p, z^p f(x_1)^{(1-p)/2}).$$

If we now come back to C given by $y^2 = x^p - x$, then S has many more rational curves, for instance $x_1 = \sum_{i=0}^{n-1} t^{p^i}$, $x_2 = \sum_{i=0}^{n-1} (at)^{p^i}$, $z = b(t^q - t)$. One may ask whether they are also related via endomorphisms of S. The answer turns out to be "yes" in a very strong sense:

Proposition 4.5. *The surface $S : z^2 = (x_1^p - x_1)(x_2^p - x_2)$ is unirational.*

First a lemma:

Lemma 4.6. *For an arbitrary polynomial A, the variety $x^{p+1} + y^{p+1} = A$ is birationally isomorphic over $\overline{\mathbb{F}_p}$ to $Av^{p+1} = u^p + u$.*

Proof. Let $c \in \overline{\mathbb{F}_p}$ satisfy $c^{p+1} = -1$. Define $z = x - cy$, so $x = z + cy$. Substituting gives $A = z^{p+1} + c^p y^p z + cyz^p$. Now divide by z^{p+1} and let $v = 1/z$, $u = cy/z + b$, where $b^p + b = 1$, and the lemma follows. □

Proof of Proposition 4.5. It was proven by Serre that the Fermat surface $x^{p+1} + y^{p+1} = t^{p+1} + 1$ is unirational (see a general result by Shioda in [Shioda], Proposition 4.2). The proof is a change of coordinates just as above. Apply Lemma 4.6 with $A = t^{p+1} + 1$, to conclude that the Fermat surface is birationally isomorphic to $(t^{p+1} + 1)v^{p+1} = u^p - u$. Let $w = tv$ so the last surface is $w^{p+1} + v^{p+1} = u^p + u$. Apply Lemma 4.6 again with $A = u^p + u$ and get that the last surface is birationally isomorphic to $r^{p+1}(u^p + u) = s^p + s$. If $\gamma \in \overline{\mathbb{F}_p}$ is such that $\gamma^{p-1} = -1$, then this last surface maps to S by $z = r^{(p+1)/2}/(u^p + u)$, $x_1 = \gamma u$, $x_2 = \gamma s$. □

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