Title. Beurling type theorems for polynomial subalgebras of $H^2$

Abstract:

Let $A$ be a finitely generated subalgebra of $\mathbb{C}[z]$ and $A_0$ be the collection of all elements of $A$ that are zero at the origin. We consider $H^2$ as a module over $A$ and shall look at the structure of closed $A$-submodule’s of $H^2$.

In the case $A=\mathbb{C}[z]$, the $\mathbb{C}[z]$ module structure of $H^2$ is well known from Beurling’s theorem. Any $z$-invariant subspace $M$ of $H^2$ is in the form $M=g \cdot H^2$, where $g$ is an inner function. For the purpose of application to operators elliptic in a half cylinder, the vector version of Beurling’s theorem was obtained by Peter Lax. It is natural to ask what happens when $A$ is not $\mathbb{C}[z]$ but some finitely generated subalgebra of $\mathbb{C}[z]$?

Theorem.

Let $A$ be generated by $p_1,...,p_n$ such that

1) $\deg p_1,..,\deg p_n$ have maximum common divisor = 1
2) $|p_1'(z)| + \ldots + |p_n'(z)| > 0$.

Let $P$ be the map from $\mathbb{C}$ to $\mathbb{C}^n$ by $z \rightarrow (p_1(z),...,p_n(z))$ and let $\beta =$ Betti number of the image of $P(\Delta)$. Then:

$$\beta \leq \sup_M \dim(M \odot A(M \odot A_0M)) \leq m(\beta + r)$$

where $M$ runs over all finitely generated $A$-submodules of $H^2$, $r$ is the number of self-tangencies of $P(\Delta)$ and $m$ is a constant depending on the type of self-tangencies.

Corollary (to the proof).

Any $A$-submodule of $H^2$ is finitely generated.